Mathematical Problem Solving Seminar
Opening Comments

Thanks so much for taking this on! I’ve always found it a joy to work with the tutors because they are so eager to dig into ideas and explore problems. I hope you’ll find this to be the case too!

There are seven sessions here. For the past 4 years it has been the norm to meet twice a month (approximately every other week) during the semester for an hour and a half each time. I set the time based on my schedule but also on what feels like it will be a reasonably slow time for the tutoring center, as math tutoring does shut down while the tutors are out for training. For me Thursday afternoons from 3:00 to 4:30 worked best, but you’ll need to find what works best for you and them. One semester I did offer two meetings for each session in order to accommodate different tutor class schedules, but I found that that didn’t work well for me, as it was often the case that 1 or 2 or no tutors at all showed up on the second offering; also it didn’t allow for as much collaboration. My solution was to choose a time that worked best for me and them (best I could tell) and then to have tutors who could not make it to the meeting visit me during office hours between sessions - having worked on the handouts on their own ahead of time - and the spending 15 minutes or 20 minutes discussing a few problems together.

I’ve rewritten the materials to include as many answers as possible and also as much work as possible for each problem. Many of these are classic “problem solving” problems. A problem is different from an exercise in that with a real problem you don’t necessarily know how to start on it when first confronted with it. Exercises are what do in math classes - for instance where we show students what the quadratic formula is and then we have them do 30 problems in which they have to use the quadratic formula. Here the tutors will be learning how to be genuine problem solvers - which helps them as they are in the center with a multitude of different questions being thrown at them, and which provides opportunity for them to learn new techniques which they can then also share with students who come in - techniques such as Look for a Pattern, State and Solve a Simpler Problem, Make an Organized List, Use Logic, Work Backwards, etc.

Please note that for each session I have included a mathematician and a quote. Do point those out to students. I tried to find quotes that were inspiring or thought-provoking or that matched our theme for that session well. For each session there are meeting notes (i.e. teacher notes) and then a handout and then a problem set. Though it doesn’t always work out this way, the normal flow of things is to do the handout collaboratively or independently while together, discuss it, and then have the problem set be like homework (but they can choose how much or how little to do, and there is no expectation of them turning it in or having done all of it). Typically the tutors are very engaged in this process and end up working together on these problems between sessions.

So that’s what’s been going on, and that’s what I do, but this is, as they say “your baby” now, so if you want to use these lessons, feel free. If you’d rather replace one or more with topics of your choice (such as an introduction to stats or something else you feel the tutors could benefit from) feel free to do that too! Enjoy!!
INTRODUCTION AND SUGGESTIONS:

Given the structure and intent of this book, this lesson is intended to be the introductory lesson for a new semester, a session in which an overview is given to the seminar in general as well and where the new game for the new semester is introduced. This is why this topic, one that is very accessible, was chosen for this first meeting. The topic itself is not well-represented by a plethora of clever problems; the best one, the one which should receive a lot of focus is the classic Census Taker Problem (#1 on the handout).

It is typically the case that the handout is used during the session and the problem set is handed out at the end of the session and assigned as “homework,” but this is a very flexible arrangement. Sometimes both the handout and the problem set are used during the seminar - depending on the problems and what is desired and where serendipity leads. And, in terms of the problem set being homework, it’s more along the lines of things for the students to continue thinking about. They or the leader might want to refer to some of these problems in the next session, but it is NOT a matter of collecting and grading it nor of the participants being expected to have done every problem.

If this session is used as the first in a new semester take time first to introduce the seminar itself, with its structure and goals. As time allows go over the other problems on the handout together. SAVE ENOUGH TIME to introduce this semester’s game, Dots and Boxes. Depending on the make-up of the group, it might be a good idea to discuss the importance of play in mathematics - ideas for this would be to mention G. H. Hardy’s view of mathematics and the role now of number theory. Hardy gloried in the fact that his work was theoretical and not applied, saying, “Nothing I have ever done is of the slightest practical use.” He also said, “I have never done anything ‘useful’. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.” This may sound strange, but given that he lived through World War I and detested the military uses to which mathematics had been applied it isn’t surprising that he was pleased that his work had no application. Much of his work was in number theory, and, as so often happens with mathematics, even if an application is not evident at first it almost always ends up applying to a new situation in the future. Today use has been found for Hardy’s work in number theory; for one thing, it is has become indispensable in our lives daily for encryption of sensitive data on the internet.

There is a phrase that describes this well, “the unreasonable effectiveness of mathematics in the natural sciences.” This was the title of a 1960 mathematical paper, and it is that case that the most abstract mathematics often does turn out to have unsuspected
applications. A good example of this is knot theory. In the mid-1800s Sir William Thompson, Lord Kelvin, proposed that the structure of the atom may be that of a knotted vortex of ether (a substance thought to pervade the universe). This precipitated a tremendous amount of study of mathematical knots. As it turned out the structure of the atom is not that of a knot, but even when they had no practical reason to study knots, mathematicians kept at it simply because it was interesting in its own right. In the 1900s, as science advanced and the structure of the DNA was discovered, there was a need for understanding knotted structures, and mathematicians had laid the groundwork necessary through their play. Knot theory also has applications to the physics of string theory and to the chirality (left-handedness or right-handedness) of molecules, which is important in chemistry and in pharmaceuticals. Play is fun, and when you play with math you never know where what you discover might take us!

DOTS AND BOXES GAME:

Play begins with a rectangular grid of dots. Players take turns to draw a line joining two dots that are adjacent either horizontally or vertically (but not diagonally). If a player completes the fourth side of a box - a square of unit side - then that player writes his or her initial in that box and plays again (and must continue to play again as long as they keep forming completed boxes). When all possible connections have been made, the game ends, and the player who has initialed the most boxes wins. (There is a handout with directions and with grids for play.)

HANDOUT - COMMENTS AND ANSWERS TO SELECTED ITEMS:

1. THE quintessential and VERY clever problem that best illustrates this strategy is the ‘Census Taker Problem.’ It is a problem that seems impossible at first - as if there is not enough information, but all it takes is a willingness to list all possibilities and then to consider them according to the clues. If it is known that some people in the group have already seen this problem, they should be given the Problem Set to work on instead so that they don’t give anything away, even inadvertently - such is the nature and power of this problem. The KEY is that after hint #2 the census taker still can’t figure it out, but upon hearing a comment about the oldest he can. Really give this one time! (Answer: 2, 2, 9)

2. There are 23 triangles.

3. This is a particularly interesting problem to discuss, as there are many ways participants might go about listing the possibilities. It would be a good idea to have some volunteers share their methods. Answer: 121.

4. The last item would be returned on Christmas Day of the next year (or Christmas Eve if the year is a leap year). It will take 364 days (beginning on December 26) to return all
the gifts, as 364 is the sum of the first twelve triangular numbers:
\[ 1 + (1 + 2) + (1 + 2 + 3) + \cdots = 1 + 3 + 6 + \cdots = 364 \]
The items received most are swans and geese at 12 each \((7 \times 6 \text{ or } 6 \times 7)\).

**PROBLEM SET - COMMENTS AND ANSWERS TO SELECTED ITEMS:**

1 27 (Try listing triangles by size, and don’t miss the inverted central triangle.)

2 104 (Good luck!) One approach would be to consider an equilateral triangle in which straight lines are drawn from each vertex to the middle of the opposite side. There are 16 triangles within such an image, and the image in this problem is made of four of them, bringing us up to 64. Consider then the triangles formed when combining pieces of more than one of these smaller, component triangles.

3 There are multiple answers and multiple means of approach. A list could be made of all triplets whose sum is 12. All permutations (that do not start with zero) of those triplets could be considered; odd numbers and numbers which are not a multiple of four (i.e. whose last two digits are not a multiple of four) can be eliminated. At that point there are fourteen three-digit numbers to consider, which can be checked by division. Another approach would be to list all three-digit multiples of 24 and eliminate those that do not satisfy the conditions. A pattern soon emerges, with 192 being the first number that meets the criteria and with most numbers following being at intervals of 72: 192, 264, 336, 408, 480, 552, 840, and 912. The pattern breaks down between 624 and 840 where the difference is greater than 72. Are there other ways to approach this? From divisibility rules we know that the number is divisible by 24, which makes it even and divisible by 3 but not 9. Can other such insights help speed up the solving process?

4 Problem four is for exploration only and is an extension of this semester’s game of *Dots and Boxes*. Discussions of symmetry or parity could come from this.
Session 8: Use an Organized List - Handout

“A good stack of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.”

Paul Halmos (1916-2006)

1. A census-taker knocks on a door, and asks the woman inside how many children she has and how old they are. “I have three daughters; their ages are whole numbers, and the product of their ages is 36,” says the mother. “That’s not enough information,” responds the census-taker. “The sum of their ages is equal to the address on the house next door.” The census taker looks next door and sees the address. He still can’t figure out their ages. The mother then says, “Sorry, I need to go. I’m working on a project with my oldest child,” and she closes the door. The census taker then realizes what the ages are. What are the ages of the three daughters?

2. How many triangles are in the figure below?
3. If you add the digits in a number, how many numbers between 0 and 1000 have a digit-sum of ten? For example, one of the numbers is 334 because $3 + 3 + 4 = 10$

4. On the first day of Christmas my true love gave to me a partridge in a pear tree. Unfortunately my lease precludes owning a pet, so I decided I’d better take it back the following day (12/27). No sooner had I returned it to the Partridge Mart than I found my true love had sent two turtle doves and a partridge (yes, another) in a pear tree. My plan was to return these three gifts, one each day, for the next three days, beginning with the following day. If my true love continued giving me the appropriate gifts for each of the 12 days of Christmas, and I continued returning one gift each day, on which day would I finally return the last item? What item(s) did I receive most of?
**RULES AND INFORMATION:** Dots and Boxes (also known as Boxes, Squares, Paddocks, Pigs in a Pen, Square-it, Dots and Dashes, Dots, Smart Dots, Dot Boxing, or, simply, the Dot Game) is a pencil and paper game for two players (or sometimes, more than two) first published in 1889 by Edouard Lucas.

Starting with an empty grid of dots, players take turns, adding a single horizontal or vertical line between two unjoined adjacent dots. A player who completes the fourth side of a $1 \times 1$ box earns one point and takes another turn. (The points are typically recorded by placing in the box an identifying mark of the player, such as an initial). The game ends when no more lines can be placed. The winner of the game is the player with the most points.

The board may be of any size. When short on time, 2 boxes (created by a square of 9 dots) is good for beginners, and $5 \times 5$ is good for experts.

The diagram below shows a game being played on the $2 \times 2$ board. The second player (B) plays the mirror image of the first player’s move, hoping to divide the board into two pieces and tie the game. The first player (A) makes a sacrifice at move 7; B accepts the sacrifice, getting one box. However, B must now add another line, and connects the center dot to the center-right dot, causing the remaining boxes to be joined together in a chain as shown at the end of move 8. With A’s next move, A gets them all, winning 31.

(Text and image copied from Wikipedia)
DOTS AND BOXES

Here are some grids of various sizes for you to play “Dots and Boxes” on. Can you find strategies for good play? Do strategies change for grids with even vs. odd dimensions?

2 × 2

```
• • •
• • •
• • •
```

3 × 3

```
• • • •
• • • •
• • • •
• • • •
```

4 × 4

```
• • • • •
• • • • •
• • • • •
• • • • •
```

5 × 5

```
• • • • • •
• • • • • •
• • • • • •
• • • • • •
• • • • • •
```

12
Session 8: Use an Organized List - Problem Set

“A good stack of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.”

Paul Halmos (1916-2006)

1. One of the many math problems floating around on the social media site Facebook during summer 2013 was the following. How many triangles are there in the figure?

![Triangle](image)

2. If you’re up for a challenge, here’s a more complicated one. How many triangles are there in this figure?

![Complex Triangle](image)

3. A three-digit number is divisible by 8. Its digit sum is 12. What is the number? (Is there more than one answer?)
4. At our meeting you were introduced to the “Dots and Boxes” game. We explored the game a bit using square-shaped playing boards, and the question was asked whether the dimensions of the board (odd-by-odd or even-by-even) made a difference in the strategy. What if the board is not square? What if it is even-by-odd? Does this effectively change strategy in any way. Try playing with the boards below and see how you do.

2 × 3

3 × 4

Make your own grid:
“Suppose aliens invade the earth and threaten to obliterate it in a year’s time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world’s best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.”

image: Frank P. Ramsey (1903-1926)
quote: Paul Erdős (1913-1996)

INTRODUCTION AND SUGGESTIONS:
This session focuses on the Pigeon Hole Principle, but it also includes an excursion into Ramsey Theory. Most seminar topics have to do with general techniques for problem solving. This is different, as the Pigeon Hole Principle is a very specific technique and only useful in a very specific (but surprisingly diverse) set of circumstances. A goal of these seminars, along with providing specific problem solving techniques, is to introduce the tutors, many of whom are future math majors, to the bigger picture of mathematics, of which Ramsey Theory is a particularly interesting topic. For this session comments and suggestions are included with the answers for both the handout and the problem set.

HANDOUT - COMMENTS AND ANSWERS TO SELECTED ITEMS:
Begin by having the students do problems 1 through 5 on the first page of the handout - finding convex polygons in a collection of 17 dots in general position (i.e. no three dots on a straight line). Note to instructor: with 17 dots we are guaranteed a convex hexagon, but we are not guaranteed convex heptagons or octagons; they might be present, but they are not guaranteed to be present.

After talking about this, move on to problem 6 and let students ponder it a bit. (One possible answer is below, but save for next time if they haven’t discovered a solution).
Discuss the “Happy Ending Problem.” This problem was a collaboration between Paul Erdős, Esther Klein and George Szekeres (“sekeresh”) to prove that there exists a number of dots $K(n)$ for any desired number of vertices $n$ such that a convex polygon with that many vertices must exist in the collection of dots. It is known that $K(3) = 3$, $K(4) = 5$, $K(5) = 9$, and $K(6) = 17$. It is also known that the value $K(n)$ for all $n$ is finite, but $K(n)$ for $n > 6$ is not known. Erdős termed this the “Happy Ending Problem” because after their collaboration Esther and George got married.

Talk through problem 7 together. Maybe act it out or discuss an idea of “worst case scenario” or maxing it out. Move on to problem 8 and let them work at this a bit. Problem 8 is a classic Pigeon Hole Principle problem. The Pigeon Hole Principle states that if $n$ items are put into $m$ containers, with $n > m$, then at least two items must be in the same container. It doesn’t guarantee exactly two. Consider a game where there are 3 chairs and 4 people and in which each person needs to be seated in a chair. What are the possibilities? Maybe all 4 sit pile up on one chair, maybe two chairs have one person and one chair has 2 people, but not matter how you split it up there are at least two people in one chair. For problem 8 find the pigeon holes by focusing on what needs to be the case for numbers to sum to a multiple of 10 (i.e. end in zero). If there is a pair of numbers such that one ends in 1 and the other in 9, that pair sums to a multiple of 10. There are 6 such situations 1 & 9, 2 & 8, 3 & 7, 4 & 6, 5 & 5, and 0 & 0. No matter how you choose seven numbers you will have seven items for only 6 containers, therefore you are forced to have such a pair that sums to a multiple of 10. Of course if two numbers end in the same digit, then you have a pair whose difference is a multiple of 10.

Work through problems 9 and 10 together. Problem 9, the six-people-at-a-party problem, is moving us towards Graph Ramsey Theory. We can use a “coloring” approach to show that there are at least 3 mutual strangers or at least 3 mutual acquaintances. Create a graph with 6 vertices, each vertex representing a person. Each edge will represent the relationship between the two people/vertices it connects. Let a solid (or red) edge represent acquaintances, and let a dotted (or blue) edge represent strangers. A triangle of all solid (or red) lines represents 3 mutually acquainted people, and a triangle of all dotted (or blue) lines represents 3 mutual strangers. Without loss of generality (wlog, begin at the upper left-hand vertex; also wlog notice that we will have at least three edges of the same type emanating from that vertex. (See figure 1.) Notice that there are three connections we could make that would create a triangle, which is what we are trying to avoid, so we use dotted (or blue) lines in two cases (figure 2), but whatever edge we use to connect the two vertices we are forced to create a triangle of one sort or the other (figure 3). Therefore, at a party with 6 people, it is the case that there are at least 3 that are mutual acquaintances or mutual strangers.
For **number 10**, let the students think about this a bit too before jumping in as an instructor. To answer this, create an equilateral triangle as below. Here our pigeon holes are colors, and the pigeons are the points. There are more pigeons than holes (i.e. points than colors), so at least 2 must be the same, and each are at a set distance $d$.

This is the only seminar for which there is an *extended handout*. It is included for fun and to allow students to go a little deeper. Spend some time talking about Bible Codes and Ramsey Theory - really emphasizing the phrase, *“Given a large enough structure you are guaranteed to find a smaller substructure.”*  

As to the quote at the top of our pages for this seminar, “The classic Ramsey problem can be phrased in terms of guests at a party. What is the minimum number of guests that need to be invited so that either at least three guests will all know each other or at least three will be mutual strangers?” [The answer in this case is 6, and this is the flip-side of problem 9 for us.] “Suppose we want not a threesome but a foursome who either all know each other or are all mutual strangers. How large must the party be? Erdös and Graham and their fellow Ramsey theorists have proved that 18 guests are necessary and enough. But raise the ante again, to a fivesome, and no one knows how many guests are required. The answer is known to lie between 43 and 49. That much has been known for two decades, and Graham suspects that the precise number won’t be found for at least a hundred years. The case of a sixsome is even more daunting, with the answer known to lie between 102 and 165. The range grows wider still for higher numbers.”  

[from pp. 52-53 of *The Man Who Loved Only Numbers: The Story of Paul Erdös and the Search for Mathematical Truth* by Paul Hoffman]

**PROBLEM SET - COMMENTS AND ANSWERS TO SELECTED ITEMS:**

Do encourage students to focus on the Problem Set more than usual for next time, as we might want to pick up on these problems next time. Don’t give hints - the idea is that they will figure out that they will need to ‘state and solve a simpler problem,’ which is our topic for next time.

1. Though this problem seems like number 8 on the handout it is actually more subtle and complicated. It may end up needing to become a “back-burner problem” for the
semester - that is a problem that the students continue to think about for a number of weeks. That’s OK as persistence is an important quality of a problem-solver. This problem can be approached by looking at smaller numbers, but really there is a deeper number theory issue going on here. When working with a larger number such as 19 it’s easy to lose sight of the combinations of numbers that could sum to 19.

Let the 19 numbers be \( a, b, c, d, \ldots, s \). Consider 19 specific sums, as follows:

\[ a, a + b, a + b + c, a + b + c + d, \ldots a + b + c + \cdots + s \]

Note that when numbers are divided by 19 the only possible remainders are 0, 1, 2, \ldots 18, a total of 19 possibilities. Consider two cases, either a number, say \( a \) has a remainder of 0 (in which case we are done), or there is not a number with a remainder of zero when divided by 19. In this case there are only 18 possible remainders left for our list of sums, but there are 19 different sums under consideration, so, by the ‘Pigeon Hole Principle, two of them must have the same remainder. Let’s say that these two sums are

\[ a + b + c + d \]

and

\[ a + c + d + e + f + g + h \]

Since these have the same remainder when divided by 19 their difference must be a multiple of 19. For instance, the first number will be of the form \( 19n + r \) and the second number will be of the form \( 19m + r \). If we subtract we get:

\[ [19m + r] - [19n + r] = 19m - 19n = 19(m - n) \]

which is a multiple of 19. The difference between our two numbers shown above is:

\[ (a + b + c + d + e + f + g + h) - (a + b + c + d) = e + f + g + h \]

So, the sum \( e + f + g + h \), which is some sum of some set of 19 numbers is divisible by 19. (If you have not already done so, do try this with a much smaller number and see what happens in that case.)

2. This problem also involves PHP, but it’s tricky to figure out what the ‘holes’ and what the ‘pigeons’ are. The holes are the distances, of which there are 9 possible, and the pigeons are the meals, of which there are 10.

3. Draw a square and connect the midpoints of the sides (vertically and horizontally only); then draw the two diagonals. Notice that the distance across the diagonal is \( \sqrt{2} \). By placing the points in the 4 smaller squares we can keep a distance more than \( \frac{\sqrt{2}}{2} \), but once we place a 5th point in the square it must be closer than \( \frac{\sqrt{2}}{2} \) from another point.

4. More than any other problem in this problem set this one leads us to our next topic, which is State and Solve a Simpler Problem. Don’t tell the students this in advance however. There are 222 such paths from \( A \) to \( B \). Rather than trying to count each one,
counting the number of paths to points closer to A, in a systematic way, can yield a clear pattern that can then be used to answer the question. This pattern is even clearer in question 1 of the handout for the next session; in that case there is no hole in the middle of the grid, and the pattern that emerges is that of Pascal’s Triangle. In the next session you may want to talk about that problem first and then come back to this one. (It is also possible to do both of these problems using combinatorics from the beginning rather than stating a simpler problem and looking for a pattern.)
“Suppose aliens invade the earth and threaten to obliterate it in a year’s time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world’s best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.”

image: Frank P. Ramsey (1903-1926)
quote: Paul Erdős (1913-1996)

The following dots are in *general position* (i.e. no three lie on a straight line).

1) If it exists, find a convex quadrilateral in this cluster of dots.
2) If it exists, find a convex pentagon in this cluster of dots.
3) If it exists, find a convex hexagon in this cluster of dots.
4) If it exists, find a convex heptagon in this cluster of dots.
5) If it exists, find a convex octagon in this cluster of dots.
6) Draw 8 points in *general position* in such a way that the points do not contain a convex pentagon.

7) In my family there are 2 adults and 3 children. When our family arrives home how many of us must enter the house in order to ensure there is at least one adult in the house?

8) Show that among any collection of 7 natural numbers there must be two whose sum or difference is divisible by 10.

9) Suppose a party has six people. Consider any two of them. They might be meeting for the first time - in which case we will call them mutual strangers; or they might have met before - in which case we will call them mutual acquaintances. Show that either at least three of them are mutual strangers or at least three of them are mutual acquaintances.

10) Points in the mathematical plane are each colored with one of two colors: red or blue. Prove that, for a given distance $d$, there always exist two points of the same color at the distance $d$ from each other.
Session 9: Pigeon Hole Principle and Ramsey Theory - Extended Handout

In the ‘opener’ for our seminar today you were asked to find vertices of convex polygons within a group of dots in ‘general position.’ This problem comes from Ramsey Theory (which is related to the Pigeon Hole Principle). Frank P. Ramsey was a mathematician, philosopher and economist in the early twentieth century who made significant contributions to all three fields before his untimely death at age 26. Ramsey Theory, which has become an entire branch of mathematics in its own right comes from a small lemma that he proved as a bit of an aside in a paper on philosophy.

Image from *Bible Code II: The Countdown* by Michael Drosnin

The basic idea of Ramsey Theory is that if you are **given a large enough structure you are guaranteed to find a smaller substructure.** That’s the idea in the dots problem at the beginning of our work: given enough dots, can you find a certain polygon, and how many dots must there be in order to ensure that such a polygon can be found? This is called the ‘Happy Ending Problem.’

People can often be overly impressed or inordinately enthusiastic when patterns, especially patterns that seem meaningful to them, arise from seeming chaos. An interesting example of this is the “Bible Codes” written about by reporter and author Michael Drosnin. Figure 1 shows a passage in Hebrew scriptures in which a search has been done by examining letters appearing at a certain fixed interval from each other - for instance every fifth letter or every sixtieth letter. In this figure we see Hebrew letters that give us the words/phrases ‘twin,’ ‘towers,’ ‘it knocked down,’ ‘twice,’ and ‘airplane.’ This seems to be a prediction of

---

1It was called the ‘Happy Ending Problem’ by Paul Erdős who worked on it with mathematicians Esther Klein and George Szekeres. The collaboration resulted not only in a proof that there does exist a number $K(n)$ for any $n$, where $n$ is the number of vertices in the polygon you are looking for, but it also in George and Esther’s marriage.
the 9/11 terrorist attack. This is striking, given that the Torah is thought to have been revealed to Moses over 3000 years ago!

And this is not the only such ‘prophecy’ contained in Hebrew scripture. Assassinations of famous people, such as prime ministers, have been found in this way, as have events such as the start of the Gulf War and the use of the atomic bomb. It has been claimed by Michael Drosnin that because of the special nature of the scripture that no other book contains such predictive power. Of course many people are critical of his findings. When he was interviewed by Newsweek Magazine, Drosnin said, “When my critics find a message about the assassination of a prime minister encrypted in Moby Dick, I’ll believe them.”

Well, guess what? A group of mathematicians and computer scientists took him up on his challenge and used this same technique on Moby Dick, and in Figure 2 you see one of their findings, which is a purported prediction of the death of Princess Diana - including not only her name and Dodi’s name (both twice!), but also “mortal in the jaws of death,” “road,” “skid,” and even the name of the driver Henri Paul!

Also in Moby Dick were found ‘predictions’ of the assassinations of Indira Ghandi, Lebanese President Rene Moawad, Leon Trotsky, Rev. Martin Luther King Jr., Yitzhak Rabin, John F. Kennedy, and Abraham Lincoln! Similar results would be the case if we searched any tome, such as War and Peace or Gone with the Wind.

I write this up for you not, of course, intending to downplay the sacredness of scripture, but rather to underscore the fact that Ramsey Theory is what is operative here - that given a large enough structure you are guaranteed to find a smaller substructure.
Speaking of the assassination of Presidents Lincoln and Kennedy, there are some rather startling ‘coincidences’ there.

- Both were elected president in ’60 (i.e. 1860 and 1960).
- Both were elected to the House of Representatives in ’46.
- Both were succeeded by their VPs who both had the last name Johnson and who both were born in ’08.
- Lincoln was shot in Ford’s theater, and Kennedy was shot in a Lincoln automobile made by Ford.
- Lincoln had a secretary named Kennedy, and Kennedy had a secretary named Lincoln.
- Both were assassinated by Southerners and both were succeeded by Southerners.
- Both assassins were born in ’39.
- Both assassins were known by their three names (John Wilkes Booth and Lee Harvey Oswald).
- Neither assassin stood trial - each being killed before a trial could be held.
- Both presidents were shot on a Friday; both were shot in the head.
- Booth ran from a theater and was caught in a warehouse; Oswald ran from a warehouse and was caught in a theater.
- A week before Lincoln was shot he visited Monroe, Maryland, and a week before Kennedy was shot...um...

In their book *Coincidences, Chaos, and all that Math Jazz* authors Edward Burger and Michael Starbird ask whether we should expect such coincidences by random chance or if the existence of such parallels is an eerie, supernatural message from the great beyond. They go on to state that given such a high profile event as the assassination of a popular president, such scrutiny is given to the even and to their lives that hundreds of thousands or perhaps millions of facts are known. This means there is a seemingly endless collection of data that we can consider. “How many people are associate with Lincoln and Kennedy? How many dates are associated with their lives? How likely is it that there would be no coincidences of dates and names among this blizzard of possibilities? The likelihood that there would be no coincidences is essentially zero.”

Since they lived about 100 years apart, the chances are even greater that the last 2 digits of dates relating to their lives will match up. What about the dates that don’t match? Lincoln was born in ’09, and Kennedy was born in ’17. Kennedy’s wife was born in ’29, and Lincoln’s wife was born in ’18. Kennedy was married in ’53, and Lincoln was married in ’42. Lincoln died in ’65 and Kennedy died in ’63. In fact MOST of the Lincoln-Kennedy dates do NOT coincide. “One of the keys to putting coincidences into perspective is to
realize that we haven’t decided what type of coincidence we are seeking before we happen to witness it.”

Additionally, how many thousands of people are associated with someone so prominent as a president? “The coincidences about Lincoln and Kennedy are notable and known because Lincoln and Kennedy are famous. However, if we took any two ordinary Joes or Janes and delved as deeply into their lives as historians and journalists have delved into the lives of Lincoln and Kennedy, we would find amazing coincidences there too. Coincidences do not arise because people are prominent. They arise when we ask so many questions that the vast numbers of opportunities make the chance for coincidences overwhelming.”

Coincidences are tremendously fun and attention-grabbing, but it bears repeating: **given a large enough structure you are guaranteed to find a smaller substructure.**

---

2 All quotes, and most information for the Kennedy/Lincoln portion of this document, are taken from *Coincidences, Chaos, and all that Math Jazz* by Edward B. Burger and Michael Starbird.
Session 9: Pigeon Hole Principle and Ramsey Theory - Problem Set

“Suppose aliens invade the earth and threaten to obliterate it in a year’s time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world’s best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.”

image: Frank P. Ramsey (1903-1926)
quote: Paul Erdős (1913-1996)

1. Show that if you have 19 positive integers, then either one of them is a multiple of 19 or a sum of several of them is a multiple of 19.

2. Ten people are sitting around a circular table at a restaurant. When their food is served, they notice that no one was handed the correct dish. Show that it is possible, by simply rotating the table, to ensure that at least 2 people have the correct dish.
3. Show that if five points are chosen in or on the boundary of a unit square that at least two of them must be no more than $\frac{\sqrt{2}}{2}$ units apart.

4. If one must always move upward or to the right on the grid below, how many paths are there from $A$ to $B$. 

![Grid Diagram](image-url)
“If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.”

John von Neumann (1903-1957)

INTRODUCTION AND SUGGESTIONS:

If participating in the AMATYC SML Competition (or another competition) the next meeting might used as a time to administer the contest rather than as a seminar time, unless it is decided to hold the competition at another time such as a Friday or Saturday or evening or whatever. The contest should be optional. If a competition is being held, students should be given a reminder about that. Information should be given for preparation for those who wish to compete. The meeting after the competition could include a reflection session on the competition and on our game for the semester, so remind them of that as well so that they are thinking ahead and working on the game strategy.

The topic of this session, State and Solve a Simpler Problem, is one of the most powerful problem solving techniques that exists. Often simply using smaller (or easier) numbers helps us to see to the heart of the problem. Combining this technique with looking for a pattern (i.e. trying numerous examples with smaller numbers) is extremely powerful. It may be helpful to look back at the previous problem set while moving into this session - or it may be best to simply begin with problem #1 of the handout for this session and then to look back at problem #4 from Problem Set 9.

After considering and discussing problem #1 give participants time to work on problems 2, 3, 4 and 5 of Handout 10. After they’ve had sufficient time, discuss these and discuss also A and B at the end of that worksheet, which is a suggestion to the tutors attending of how to share this strategy effectively with students they are working with. (NOTE: in the past when the session included tutors and non-tutors, the non-tutors always gravitated towards problems A and B while skipping problems 2 through 5. I think they did this because they are conditioned to work with that sort of problem. Also instead of trying to make the problems simpler they just dug into them like robots, using the numbers they were given. The point of these problems is that they are much easier if you use numbers such as 10 and 1/2 for problem A and if you ditch the fractions and also use simpler numbers for problem B, and once you’ve figured out how to set it up you can use the same set up with the “harder” numbers.)

IF the competition will take the place of our next meeting, then distribute Handout 11 and Problem Set 11 (with answers) so that participants will be able to prepare for the competition. IF we will be meeting instead and holding the competition on another day, THEN hold onto the documents for session 11 until that time and use them as a general intro to competitions.
HANDOUT - COMMENTS AND ANSWERS TO SELECTED ITEMS:

1. **Answer: 1297.** If you try counting all such paths it will become nearly impossible to keep track of, and there is much room for error given that you would have to count 1297 things! Start smaller - begin with the $1 \times 1$ grid only in the lower left corner, then expand your counting to the lower left-hand $2 \times 2$ grid. Fill in the number of paths not just to the opposite corner but also to every intersection. You will find that the intersections along the bottom line all have 1 path for getting there - similarly the intersections along the left-hand vertical line. What will develop using this method is a slice of Pascal’s Triangle. There are a couple of ways to think about why this is. One thing to think about is how you get into each vertex; you have to come from below or from the left, so the number of ways to get into any vertex is the sum of the two numbers below and to the right (and this sort of addition is also what gives us Pascal’s Triangle). Additionally, Pascal’s Triangle is composed of combinations, and as we do this we are literally counting the number of ways to do something. It is possible to use a strictly combinatorics approach to this, which is interesting to note, but remember that the focus of this sessions is to make the problem simpler.

2. Students may or may not recognize this immediately as a power of 2, but if they start finding factors this should become obvious early on. Rather than trying to find all the factors of this number use smaller powers of 2 and see if there is a pattern in the sum of their divisors. For instance the divisors of $2^2 = 4$ are 1, 2, and 4. The sum of 1, 2, and 4 is 7 (which is one less than the next power of 2). Also the divisors of $2^3 = 8$ are 1, 2, 4, and 8; their sum is 15 (which is one less than the next power of 2). This is true for sums of factors of powers of 2 for all powers of 2, so the sum of the factors of 2048, which is $2^{11}$ is $2^{12} - 1$ or 4095 (or just look at it as $2 \times 2048 - 1$).

3. The number here is large. Start with smaller numbers and look for patterns. This might be one to let students work on between now and next time (i.e. a "back-burner problem). The number of factors a number has is related to its prime factorization. All of the factors of a number are also related to factors of the original number. Here too let’s think about a smaller number, say 12. The prime factorization of 12 is $2^2 \times 3^1$. The factors of 12 are 1, 2, 3, 4, 6, and 12. Consider how the factors of 12 break down in terms of powers of 2 and 3 (the prime factors of 12):

\[
egin{align*}
1 &= 2^0 \times 3^0 \\
2 &= 2^1 \times 3^0 \\
3 &= 2^0 \times 3^1 \\
4 &= 2^2 \times 3^0 \\
6 &= 2^1 \times 3^1 \\
12 &= 2^2 \times 3^1
\end{align*}
\]

Given that in the prime factorization of 12 the 2 is raised to the second power, we have 3 choices of exponents to put on two in order to create factors of 12; those choices are 0, 1, 2. Given that in the prime factorization of 12 the 3 is raised to the first power, we have 2 choices of exponents to put on 3 in order to create factors of 12; those choices are 0 and 1. Since we have 3 choices of exponent for the first prime factor and 2 choices of exponent for
the second prime factor, we have $3 \times 2 = 6$ factors of 12. In the case of the problem given, we have $(1 + 1)(2 + 1)(5 + 1) = 36$ factors.

4. Here too starting with something smaller and looking for a pattern is the key. Instead of considering a $1 \times 12$ rectangle, consider $1 \times 1$ then $1 \times 2$ then $1 \times 3$. Look for a pattern in the results you are getting. If in each case you determine your answer by adding the number of rectangles of each size that you get you will notice that in each case you have $1 + 2 + 3 + \cdots$. In other words the answer is a triangular number. For a $1 \times 12$ rectangle the result is 78. For a $1 \times n$ triangle the result can be found using \( \frac{n(n + 1)}{2} \).

5. This is a problem that may end up needing to be a “back-burner problem,” and if so that’s OK, since problem solving is all about persistence. The number 2 has two partitions: 1 + 1 and 2. The number 5 has six partitions 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 1 + 3, 1 + 4, 2 + 3, and 5. If we go much further the number of partitions, \( p(n) \), begins to grow very rapidly \( p(10) = 42, p(50) = 204226, \) and \( p(100) = 190569292 \). Obviously we cannot find all the partitions of the number 500. Here too look at smaller numbers, consider products of their partitions and look for a pattern. For the number 500 the partition with the greatest product is the one containing 166 threes and a two. The product itself is approximately 3,185,358 \( \times 10^{79} \).

**PROBLEM SET - COMMENTS AND ANSWERS TO SELECTED ITEMS:**

1. There are 8 arrangements. This is probably easy enough to draw out and do by “brute force,” but for the answer to the general case you will want to begin with 2 dominoes then three then four and so on and look for a pattern. The pattern that will emerge is that the answer will be \( F_{n+1} \) where \( F_n \) is the \( n^{th} \) Fibonacci Number and where \( n \) is the number of dominoes you are working with.

2. Here too, state and solve a simpler problem. Instead of taking the square root of this number, take the square root of 100 and of 10,000 and of 100,000. The pattern that emerges is that the square root of a power of ten that has an even number of zeros will have half as many zeros as the radicand. So the answer is 50.

3. With a number this large you definitely want to try smaller examples first. How many subsets are there for a set with 2 elements, for a set with 3 elements, for a set with 4 elements? The result is that the number of subsets a set has is \( 2^n \) where \( n \) is the number of elements in the original set. (This is because in each subset each original element is either put in or left out, since each has 2 possibilities there are \( 2^n \) possible subsets.

4. This problem is nearly identical to problem #5 of the handout for this session. The answer is that it is the partition that includes 658 threes and one two. A follow-up question would be how this can be proven!
1. If one must always move upward or to the right on the grid below, how many paths are there from $A$ to $B$.

2. What is the sum of the positive divisors of 2048?

3. How many positive divisors does the number $11 \times 13^2 \times 17^5$ have?

4. The figure below is a $1 \times 12$ rectangle. How many distinct rectangles of any size are in this rectangle? How many distinct rectangles of any size are in a $1 \times n$ rectangle?
5. Natural number *partitions* of a given number are the sets of sums of natural numbers that add up to that number. For instance, some of the natural number *partitions* of 10 are 1 + 4 + 5 and 1 + 1 + 1 + 2 + 2 + 3 and 3 + 7. Consider the natural number *partitions* of 500. For which partition is the product of all the terms the greatest?

6. **SOMETHING TO THINK ABOUT:** The strategy of stating and solving a simpler problem is a very powerful one across all levels of mathematics. It often gives us insight into the larger problem that allows us to solve it. Think about how a teacher could use this strategy for helping students in arithmetic or pre-algebra to better understand how to work the following problems. First think about what it is that’s going to be challenging to the students, and then consider how making the problem simpler will yield insight.

A) A plastic container holds \(\frac{3}{4}\) gallons. How many gallons does it contain when it is \(\frac{3}{4}\) full of a cleaning chemical?

B) The weight of a cubic foot of sandstone rock is approximately \(2\frac{13}{20}\) the weight of a cubic foot of water. If a cubic foot of water weighs approximately \(62\frac{1}{2}\) pounds, and a sandstone rock contains 5800 cubic feet, what is the approximate weight of the rock?
Session 10: State and Solve a Simpler Problem - Problem Set

“If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.”

John von Neumann (1903-1957)

1. Below is one way of partitioning a $2 \times 5$ rectangle with dominoes. (Assume dominoes are $1 \times 2$.) In how many ways can this be done? (Extension: Explore in general how many ways there are of covering a $2 \times n$ rectangle with dominoes.)

2. A googol is a 1 followed by 100 zeroes. How many zeroes are in the square root of a googol?

3. Given a set with 1,234,567 elements, how many subsets does it have?

4. Modified from the 1976 International Mathematics Olympiad: Determine the largest number which is the product of positive integers whose sum is 1976.
“Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.”

*Alfred North Whitehead (1861-1947)*

**INTRODUCTION AND SUGGESTIONS:**

We may or may not meet for this ‘session.’ It depends on whether we hold the competition during our regular meeting time. If we do hold the competition during this meeting time, these documents can be made available to students ahead of time to use for practice — either by posting them online and letting students know they are there or by handing them out at the previous session. They contain problems from old SMC competitions.

If you do meet you can use this as a training time for the competition or just as a pseudo-competition allowing students to work in groups and saving some time at the end to present their favorite problems and approaches to the group. If you do cover this together you will probably get through the entire handout and maybe have some time to begin the Problem Set. Be sure to send the answers to the problem set home with the students so that they can check themselves as they practice. Answers (but not hints) to the Problem Set (and Handout) are included on the page after the Problem Set.

Students can, of course, access all the old competitions and answers online by googling AMATYC SML Competitions (or by going to the Problem Solving page of my website), but here I’ve selected a few problems that seem particularly approachable, that have nice features to them, that illustrate some of the techniques we’ve looked at this semester and last semester, and that represent a variety of problem-types.

As is mentioned on the handout, point out to students that writers of competitions often make use of the year the competition is being held in (so students should just automatically know to prime factorize this year and to study it for any special properties before the competition). Writers also often use the letters of the name of the organization putting on the competition, in this case AMATYC or SML.

**HANDOUT - COMMENTS AND ANSWERS TO SELECTED ITEMS:**

If we meet, have these 4 problems be something the students take as a mini-competition — maybe for 12 minutes or so, which would give them an idea of the pace of the competition, which is 20 problems in one hour. They could then do some collaborations in groups, comparing what they got and how they got it, and then doing presentations for the group at the board in order to learn from each other. If there is more time, then give the problem set during the session as well.

1) **The answer is B) 20.** This comes from $(96)(95)(87) = 793440$. Why put the 7 after an 8 rather than after a 9? Strategies to use and Guess & Check and Logical Reasoning.
2) **The answer is E) 7264.** The floor function might be new notation for the students. Once you get going with this one it goes quickly. Notice that \([\log 1]\) through \([\log 4]\) equal 0, \([\log 5]\) through \([\log 24]\) equal 1, etc. It changes at each power of 5. We get:

\[
4(0) + 20(1) + 100(2) + 500(3) + 1386(4) = 7264
\]

3) **The answer is A) \(10^{1002}M\).** Students may or may not have worked with multiplication of matrices, so you may need to talk about it after they try the contest (or let a tutor who has done so come up and explain to his colleagues). Good strategies to use here are State & Solve a Simpler Problem and Look for a Pattern. In other words, find \(M^1, M^2, M^3\), etc. until you see what underlies this. Two patterns will emerge - one relating to even powers of \(M\) and one relating to odd powers of \(M\):

\[
M^n = \begin{cases} 
0 & 2 \cdot 10^{\frac{n-1}{2}} \\
5 \cdot 10^{\frac{n-1}{2}} & 0 \\
10^{\frac{n-1}{2}} & 0 \\
0 & 10^{\frac{n-1}{2}}
\end{cases}
\]

where \(n\) is odd

where \(n\) is even

4) **The answer is B) 17.** A good strategy here is to make an organized list beginning with small primes, and keep going until you find three consecutive biprimes. The list of biprimes is: 6, 10, 14, 15, 21, 22, 26, 33, 34, 35 . . . and there we have it. The largest prime factor in any of the numbers 33, 34, 25 is 17, which is the answer.

**PROBLEM SET - COMMENTS AND ANSWERS TO SELECTED ITEMS:**

1) **The answer is D) 81).** This should fall into place by carrying out the procedure carefully. It is an example of one of the easier sorts of problems on the contest

2) **The answer is C) \(\sin^2 \theta\).** This is straightforward using basic trigonometric identities - another example of one of the easier sorts of problems on the contest - though some students won't have had trigonometry yet, in which case it is not an easy problem.

3) **The answer is B) 7.** From 1 to 8 you can pair larger and smaller numbers so that they each form a sum of the square 9: \(1 + 8 = 9\), \(2 + 7 = 9\), etc. From 9 to 16 you can pair larger and smaller numbers so that they each form a sum of the square 25: \(9 + 16\), \(10 + 15 = 25\), etc. The largest differences are 8 – 1 and 16 – 9, which are both equal to 7.

4) **The answer is D) 12.** This is a nice problem to get students thinking about properties of polynomials. This polynomial is of degree 11, so it can have at most 11 \(x\)-intercepts. It is an odd function, so it must have at least one \(x\)-intercept. The answer is \(11 + 1 = 12\).

5) **The answer is C.** This can be rewritten using a law of exponents, and it falls into place pretty quickly from there, though it looks unusual perhaps. Since you are just adding another \(\log x\) as you go, this is an arithmetic sequence with common difference \(\log x\).
\[ \log x, \log x^2, \log x^3, \log x^4, \ldots \implies \log x, 3 \log x, 3 \log x, 4 \log x, \ldots \]

6) The answer is C) 2. This requires a number of elements of algebraic knowledge. Since exponents are added when bases are divided, begin by summing the geometric series 
\[ \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \]. Also, rewrite the base of the problem, 4, as \(2^2\). This gives \(2^{2 \log_2 2^2}\). Using laws of exponents, the two in front of the logarithm becomes an exponent of \(2^2\) leaving \(2^{\log_2 2}\), which is 2.

7) The answer is E) 9. Many interesting things can be brought up with this problem (though it may be that the students notice right away that 10 and 9 will work for \(a\) and \(b\)). Square numbers are the sums of odd numbers:
\[
\begin{align*}
1 + 3 &= 4 \\
1 + 3 + 5 &= 9 \\
1 + 3 + 5 + 7 &= 16 \\
1 + 3 + 5 + 7 + 9 &= 25 \\
1 + 3 + 5 + 7 + 9 + 11 &= 36
\end{align*}
\]
Here is a “proof without words” showing visually what is going on. Each new “shell” added to create the next square consists of a number of items equal to the next consecutive odd number than the one added previously.

So, one idea would be to find the two consecutive squares which are 91 spaces apart and to try those for \(a^2\) and \(b^2\). Those two squares are 2025 and 2116. This violates the restriction in the second part of the problem that \(a^2 + b^2 < 1000\). So we have to find squares that are 91 spaces apart but are not consecutive squares. This can be done by looking for 3 consecutive odds that add to 91, and when that doesn’t work try for 4, etc. A pattern emerges in these formulas (i.e. \(n + (n + 2) + (n + 4) + \cdots = 91\), and it turns out that \(n = 7\) (the first of a series of seven odd numbers added to a square to get the next required square), so the first square is 9 and the second square is 100 because 
\[9 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 9 + 91 = 100.\]

8) The answer is E) 77.

9) The answer is B) \(\ln e^x\). Consider restrictions to domain here in each case.

10) Listing is a possibility. Sometimes these can be done well using a table in a TI calculator.
Session 11: Competition Problems - Handout

“Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.”

Alfred North Whitehead (1861-1947)

All problems on this handout are taken from previous American Mathematics Association of Two Year Colleges (AMATYC) Student Math League Competitions. Notice that in many of the problems you see the letters AMTYC used and numbers that are ‘recent’ years (such as 2010) being used. This is something competition writers often do, so one thing you should do as you prepare is to be very familiar with the numerical properties of the current year. Before the day of the competition make sure you are familiar with the prime factorization of the current year and be aware of whether is has any special properties, such as being a perfect square, a triangular number, a Fibonacci number, etc.

1. In the expression \((AM)(AT)(YC)\), each different letter is replaced by a different digit 0 to 9 to form three two-digit numbers. If the product is to be as large as possible, what are the last two digits of the product? 
   SML
   A. 20  B. 40  C. 50  D. 60  E. 90

2. Let \(\lfloor x \rfloor\) represent the greatest integer \(\leq x\). Find \(\sum_{n=1}^{2010} \lfloor \log_5 n \rfloor\) 
   SML
   A. 7256  B. 7260  C. 7262  D. 7263  E. 7264

3. If \(M = \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}\) and \(N = \begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}\), find \(M^{2005}\). 
   SML
   A. \(10^{1002}M\)  B. \(10^{1002}N\)  C. \(10^{2004}M\)  D. \(10^{2004}N\)  E. \(10^{2005}M\)

4. Call a positive integer \textit{biprime} if it is the product of exactly two distinct primes (thus 6 and 15 are biprime, but 9 and 12 are not). If \(N\) is the smallest number such that \(N\), \(N + 1\), and \(N + 2\) are all biprime, find the largest prime factor of \(N(N + 1)(N + 2)\). 
   SML
   A. 13  B. 17  C. 29  D. 43  E. 47
“Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.”

*Alfred North Whitehead (1861-1947)*

1. Define the operation $\Delta$ by $a\Delta b = ab + b$. Find $(3\Delta 2)\Delta(2\Delta 3)$.

   A. 72  B. 73  C. 80  D. 81  E. 90

2. $(\tan \theta - \sin \theta \cos \theta)/(\tan \theta)=$

   A. $\sin \theta$  B. $\cos \theta$  C. $\sin^2 \theta$  D. $\cos^2 \theta$  E. 1

3. Sixteen students in a dance contest have numbers 1 to 16. When they are paired up, they discover that each couple’s numbers add to a perfect square. What is the largest difference between the two numbers for any couple?

   A. 5  B. 7  C. 10  D. 12  E. 14

4. The polynomial $P(x) = a_0 x^{11} + a_1 x^{10} + \cdots + a_{11}$ ($a_0 \neq 0$) has at most $m$ number of $x$-intercepts and at least $n$ number of $x$-intercepts. The sum $m + n$ is

   A. 9  B. 10  C. 11  D. 12  E. 13

5. The sequence $\log x, \log x^2, \log x^3, \log x^4, \ldots$ is best described as which of the following?

   A. geometric with common ratio $\log x$  B. geometric with common ratio $x$
   C. arithmetic with common difference $\log x$  D. arithmetic with common difference $x$
   E. neither arithmetic nor geometric
6. The value of $4\log_2\left(\frac{2^1}{4}\cdot\frac{2^1}{8}\cdot\frac{2^1}{16}\ldots\right)$ is

A. 1  B. $\sqrt{2}$  C. 2  D. $2\sqrt{2}$  E. 4

7. Suppose $a^2 - b^2 = 91$ ($a$, $b$ integers). If $n = a^2 + b^2 < 1000$, find the units digit of $n$.

A. 1  B. 3  C. 5  D. 7  E. 9

8. A set of seven different positive integers has mean and median both equal to 20. What is the largest possible value this set can contain?

A. 65  B. 67  C. 71  D. 73  E. 77

9. Which of the following is the identity function $f(x) = x$ for all real numbers?

A. $e^{\ln x}$  B. $\ln e^x$  C. $\sin(\arcsin x)$  D. $\arctan(\tan x)$  E. $\sqrt{x^2}$

10. The equation $a^5 + b^2 + c^2 = 2011$ ($a$, $b$ and $c$ positive integers) has a solution in which two of the three numbers are prime. Find the value of the non-prime number.

A. 38  B. 40  C. 42  D. 44  E. 46
Session 11: Answers to Handout and Problem Set

Handout:
1) A  2) E  3) A  4) B

Problem Set:
1) D. 81  2) C. $\sin^2 \theta$  3) B. 7  4) D. 12  5) C.
6) C. 2  7) E. 9  8) E. 77  9) B.
10) Listing is a possibility. Sometimes these can be done well using a table in a TI calculator.
Session 12: Dots and Boxes Game and Competition Reflection - Meeting Notes

“Go down deep enough into anything and you will find mathematics.”

Charles Schlicter

INTRODUCTION AND SUGGESTIONS:

Decide on the structure of this seminar depending on the semester and how much time participants have been able to put into thoughts about the Dots and Boxes game. The day can be used as a Dots and Boxes tournament. Grids are available on page 2 of the handout. All are $5 \times 5$ so that there cannot be ties. Some good issues to talk about are what the winning (or helpful) strategies are, whether or not you want to go first, what happens if the grid is a rectangle that is not a square, if it is always possible to force a tie in an even by even grid, what happens when you play with more than two people, etc.

Also choose problems from the most recent competition to be used if students are interested in going in that direction or if they haven’t done much with Dots and Boxes or if, for some reason, spending the full session time on gaming doesn’t feel right. OR the competition problems can be used as time-fillers during the competition if some participants finish rounds earlier than others!! Start our time with sign ins and a round robin tournament (the instructor can be in or out depending on the parity of people present). The winners will play on the board in front of the whole group (for prizes?). Those who aren’t playing can work on the competition problems. After the winners play, discuss strategy (give out awards?), and then talk about the competition problems in whatever time is left.

PROBLEM SET - COMMENTS AND ANSWERS TO SELECTED ITEMS:

The Problem Set also has dots and lines, but used differently. The next session assumes that students have not taken finite math, but some may have. The Problem Set is intended to set us up for a conversation about even and odd vertices and Eulerian circuits and trails. The emphasis of the next session is that of representation and modeling, and it begins with using graph theory to model the Instant Insanity game, after that the Problem Set can be looked at and the Königsberg Bridge Problem can be explored also in terms of representation and modeling.
“Go down deep enough into anything and you will find mathematics.”

*Charles Schlicter*

Our main focus today is this semester’s game, Dots and Boxes. As the quote above says, “Go down deep enough into anything and you will find mathematics.” Much mathematics that we find very important and applicable today began with curiosity and playfulness and not necessarily with an eye to application at all. Some of the best examples of this can be found in the history of number theory and in the history of knot theory. Additionally, though there are many definitions of mathematics, it is possible to look at mathematics as the biggest game in the universe. The great mathematician David Hilbert said, “Mathematics is a game played according to certain simple rules with meaningless marks on paper.”

So we will spend time on Dots and Boxes. We will probably also spend some time on some favorite problems from this year’s recently held AMATYC/SML competition, which some members of our group took part in.
Session 12: Dots and Boxes Game and Competition Reflection - Problem Set

“Go down deep enough into anything and you will find mathematics.”

Charles Schlicter

The figures on this handout involve dots (vertices) and line segments (edges). The goal is to try to trace each figure without lifting your pencil, without missing an edge, and without tracing an edge more than once (if possible!). Note whether or not you were able to trace each figure.
The figures on the previous side went from simple to complex as you worked your way down the page. The two figures at the bottom of this page are more complex yet. Trying to determine if a figure is traceable can become very tedious if your approach to determining this is to try over and over and over until you find a path or until you decide to give up.

One item to notice with each figure is how many edges emanate from each vertex, and specifically whether this number is even or odd. This number is known as the degree of the vertex. In the figure below the vertex labeled \( A \) has degree 3, an odd degree. The vertex labeled \( B \) has degree 4, an even degree.

As you look back at the figures on the first side of this page, consider how many even and how many odd vertices each one has; see if you can find a pattern relating that to which figures are traceable and which are not. For the figures that are traceable, does it make a difference where you start when you try to trace them? If so, does this have anything to do with the number of odd and even vertices? Use your findings to determine (without trying to trace them) whether the figures below can be traced or not.
INTRODUCTION AND SUGGESTIONS:

NECESSARY MATERIALS: Sets of Instant Insanity cubes - enough for 1 set for every 2 to 4 participants. A copy of the handout for each individual participant; the copies need to be made as two pages rather than back-to-back, because they need to refer to both pages at the same time. (Crayons, colored pencils or colored pens would be an excellent idea!)

This session will be different from most others - a very guided ‘problem solving’ experience. It is intended to illustrate how mathematical structures can be used to model physical objects in surprising ways and how the math can then be used to solve what could otherwise be a nearly impossible problem. It is based on the puzzle called Instant Insanity, which was first marketed in 1967 by Parker Brothers but is isomorphic to puzzles that had appeared earlier. There are four cubes and four colors. The goal is to place the cubes in a row in such a way that each one of the four colors is showing exactly once on each of the $1 \times 4$ sides of the rectangular prism created.

The puzzle is called Instant Insanity for a reason - is very hard to solve by trial and error! The sets of cubes you are likely using in this session, however, are ones I have created, and I didn’t put a lot of thought into what would make them particularly hard to solve by trial and error. Because of this it is possible that participants in this session might stumble across a solution accidentally, but this is due to my hasty construction. I suggest that you introduce the puzzle, explain that we will be finding a solution using graph theory, hand out the handout, then handout the blocks, and let them mess around with them for about two minutes, and then move onto the graph theory method!

The Problem Set from the previous session involved an exploration of traversability of graphs based on number of even and odd vertices. But since the Instant Insanity project may take a long period of time (half an hour to an hour and a half depending on the
group!), the puzzle should be done first. Then the lesson can shift into a discussion of that Problem Set and of Euler graphs and circuits and, if desired, the history of graph theory, including the famous Königsberg Bridge Problem. Here the emphasis should be on modeling and representation as well.

**HANDOUT - COMMENTS AND ANSWERS TO SELECTED ITEMS:**

1) The first question is a counting (combinatorics problem). There are $41472 = 3 \times 24 \times 24 \times 24$ ways to arrange the cubes. The first cube has 3 possibilities because it has 3 axes. We won’t worry about rotations, because those will be a matter of how we stack the other cubes next to (or on top of) this one. For each of the other 3 cubes there are 24 ways for then to be arranged, as there are 6 faces that can be placed touching the previous cube laid down, and then there are 4 ways to rotate the cube once that connecting face is chosen.

2) **DIRECTIONS FOR SOLVING **INSTANT INSANITY **USING GRAPH THEORY:**

Before diving in, you may want to note that part of the difficulty of solving this puzzle is that it is three dimensional - in other words we can’t see all sides at the same time. The idea behind the method we are using is to represent each cube in a manner in which we can see all sides at once. Before jumping to representation by a graph it may be desired and helpful to show a net (a representation of the cube unfolded). A net is still quite complicated to take in all at once. Graphs allow us to simplify this. A net and a graph of the same cube are given below:

![Net and graph of a cube](image)

With the net we have something we are familiar with and can visualize more readily. What the graph shows is connections between opposite sides (the three axes). This may take some getting used to. If desired create another net and make the corresponding graph. Then move forward to doing this with the actual cubes that the students have. They will need to keep track of the cubes (first, second, third, fourth); on the handouts I’ve labeled this using letters A, B, C, D. **PLEASE** make sure that the students DO NOT WRITE ON the cubes. I have provided spaces on the first page of the handout for them to set the cubes down in order. They should leave the cubes there unless they are working with one, and when they finish with it they should set it back down where it came from. This VERY IMPORTANT in order to avoid confusion!!

Now we’re ready for the first step in actually doing the solving. On the first page of the
handout the students should draw the graph of each cube on the the sets of vertices that have been provided at the bottom of the page. They should label each set of vertices in the same way (red, yellow, blue, green as in the graph given in the image above), and they should use consistent labeling throughout their work. Each cube should be represented by three edges on the four vertices. If a cube has the same color on opposite sides, this will be represented by a loop (an edge that connects that vertex to itself).

After representing each cube on the first page of the handout, students need to copy all four of these graphs onto one graph at the top of the second page of the handout. MUCH CARE needs to be taken to make sure they label each edge according to the cube it came from. This can be done by doing something like putting a letter A on each edge that came from cube A, the letter B on each edge that came from cube B, etc. It would be even better if they had a different color pen or pencil to represent each cube - or if they also made the edges look a little different for each cube (e.g. dotted lines for all three edges coming from cube A, wavy lines for all three edges coming from cube B, etc.).

The next step is where the solving takes place, and it is the hardest step. Once this step is done we just translate it to the grid at the bottom of the page and follow the directions to put the cubes in place, and what will emerge is the solved puzzle. The step we are working on here is to take the single graph at the top of the page and split it into two graphs in the middle of the page. Each of these graphs represents opposite faces of our rectangular prism (front/back, top/bottom). Each graph will need to include one edge from each cube (because that one edge will express how that one cube is positioned). Also, each graph will need to be such that each vertex is of order 2 (i.e. has two edges associated with it). In the four images below are some examples of what each of these two graphs might looks like; this is not an exhaustive list - anything that meets our criteria of having one edge per cube and having all vertices with degree 2 is a possibility.

Typically it is not too hard to find the first graph, but each edge can only be used once. In other words, if you have already determined how to orient a cube front/back, you can’t also use those two face on the top and bottom! Once you have used an edge from the top graph on the second page, indicate in some way (using pencil lightly would be a good idea) which edges you have used and then look among those that are left for four edges that will satisfy the same criteria for the second graph in the middle of the page. Once these two graphs are completed you have solved the puzzle! It is, however, probably pretty hard to read off of the graphs, which is why the grid has been provided below to take you through a step-by-step translation process.

For the first two columns (front/back), look at the front/back graph above. Take note of the edge representing cube A, and list the colors of its endpoints - one for front and one for
back. Then move on to cube B and do the same thing. The only thing to keep an eye on is to make sure that you do not have the same color listed more than once under the same face (front or back). This is easy to avoid, as you can just switch the order for a cube if this happens. After the front and back are taken care of, go through this same process for top and bottom. Once this is completed you have a detailed description of how to put your cubes together. Follow the directions and place the cubes to see your accomplishment!!!

If you’ve had opportunity to do this a number of times it comes pretty easily, and you can solve this pretty quickly this way. If this is the first time then there are many obstacles to surmount - all of the translation and modeling and visualizing. The point is that we CAN model, and we CAN use graph theory to solve many types of problems both in the real world and in the realm of games and puzzles. Even if this was hard, it’s easier than having to try even one-tenth of the more than forty-thousand possibilities!! Just in case my directions are unclear or cumbersome, this same process can be found in Gary Chartrand’s *Introductory Graph Theory* on pages 125 to 131. There are also many pages online that can be found by doing a Google search for “solving instant instanity using graph theory.” One of these is a great video demo on YouTube that is only 6 minutes long!

**PROBLEM SET - COMMENTS AND ANSWERS TO SELECTED ITEMS:**

Problems 1 - 3 on the first page of the problem set are disguised versions of the items in the session 12 problem set. Students can represent each situation as a graph and then consider the number of even and odd vertices. The items on the back of the page are puzzles from GAMES Magazine, August 1985, p. 30. These don’t tie into a specific topic but fit our general theme of problem solving. I don’t have answers for these, but many students in the past have flown through these and found answers quickly, so they can certainly help each other.

1) This set-up, if thought of as a graph or drawn out as a graph (vertices indicating north shore, south shore and each island, edges representing each bridge), it should be noted that there are exactly two odd vertices, therefore this is traversable. The odd vertices are the north shore and the eastern-most island, so starting at either one of these will allow for such a tour to happen.

2) Each of these basically is a graph with palm trees as distractions! In the first case there are two odd vertices - one of which the monkey begins on - therefore he must end on the other one, which is the tree directly to the left of where he started. Similarly the second graph has two odd vertices, so the monkey must end up in the tree two spaces to the right of the one in which he started. The third image has all even vertices; this forms a circuit, so the monkey ends up at the same tree where he started.

3) This problem is a good exercise in modeling. Students should try solving this in a straight-forward, trial and error, manner to get a sense of it before shifting to graph theory (although if they are very familiar with graph theory they may see the answer immediately using ideas from graph theory but without even drawing a graph.) Most people find it challenging to turn this into a graph if they have not worked with graph theory before. First you need to determine what the vertices will be and what the edges will be.
Given the set-up of the problem the rooms (and the outside) should each be represented by a vertex, and each wall should be represented by an edge. For instance the top right and top left rooms will be connected by an edge going through them - that edge represents a path through that wall (even though the edge is perpendicular to the wall rather than running along with it, which may feel counter-intuitive, but the directions say that you need to find a path through the wall). One of the most common mistakes in this problem is that people will make 2 or 4 vertices to represent the outside (like a front-yard/back-yard type of idea and/or side-yard type of idea), but there is only one outside, and it should be represented by one vertex. Once the graph is drawn it can easily be determined that there are more than two odd vertices, therefore the graph is not traversable, therefore this task cannot be completed. There are 9 edges that connect to the outside, and each upper room has degree five, so that’s three odd vertices right there - not traversable).

4) I don’t have answers written up for this one, but typically at least a few students fly through this with ease and can share their answers if desired.
Session 13: Graph Theory - Handout

“Mathematics is the art of giving the same name to different things.”

Henri Poincaré (1854-1912)

Instant Insanity is a puzzle that was first marketed by Parker Brothers in 1967. It consists of four cubes, with each face painted one of four different colors (we’ll be using red, blue, yellow and green). The object of the puzzle is to stack the four cubes one on top of the other, so that on each side of the stack each cube face is showing a different color - in other words so that each of the four colors shows on each side.

QUESTION: In general, how easy would this be to solve by trial and error? In other words, how many different ways can you arrange the cubes while stacking like this?

The squares below are provided as a place for you to put your cubes so that you can keep track of them and not lose your place.

CUBE A  CUBE B  CUBE C  CUBE D

Draw a descriptive graph below for each of the cubes above.

CUBE A  CUBE B  CUBE C  CUBE D
Create a composite graph below containing all four graphs from the last page. Label each edge according to the cube it originally came from (A, B, C, or D) - or, rather than labeling, use 4 colored pencils to record which edge came from which cube.

\[
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}
\]

Now, use the edges from the complete graph above to make two sub-graphs below. Each sub-graph should have one edge from each cube (for a total of 4), and each vertex should have order 2 (which will mean that each color is used twice - once each front and back, once each top and bottom).

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

Use the information from the subgraphs above to write up a chart describing the solution. Make sure that each column has each color listed once only. Rows can have repeated colors.

<table>
<thead>
<tr>
<th>CUBE</th>
<th>FRONT</th>
<th>BACK</th>
<th>TOP</th>
<th>BOTTOM</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1) Below is the river Flos, which flows through the city Ruritania. The people of the city challenge each other to take a bridge tour by walking across each bridge exactly once and not missing any of the bridges. Is such a walk possible? If so, find it. If not explain why it is not possible.

```
```

2) A monkey made the tracks in the sand in each of the following figures by beginning at the tree marked by the arrow and moving from tree to tree as shown by the dotted lines. If the monkey did not retrace any tracks, in which tree or trees is it possible for the monkey to be hiding?

```
```

3) The rectangle below is separated into 5 ‘rooms.’ If possible find a path through each wall exactly once using a continuous curve. (There are 16 walls in this image.)
The following are ancient Japanese puzzles known as ‘Hiroimono,’ which means ‘things picked up.’ Originally the goal was to pick up stones from a Go board. Rather than picking up stones you will be asked to number the circles in the order you would have picked them up had they been stones. To do this you will decide on a starting circle and write the number 1 in it; then move left, right, up or down to a new circle and write the number 2 in it. Continue moving and filling in circles according to the following restrictions:

1. Moves may only be horizontal or vertical, never diagonal.

2. You may not pass over unfilled circles. You may, however, pass over filled circles or empty spaces between circles.

3. You may not retrace any part of your most recent move. For example, if you just moved left to right, your next move cannot be from right to left.

These may have more than one solution, but finding even one is sometimes tricky. Tip: keep an eraser handy or make multiple copies of this page. One has been done for you as an example:

---

*Recreated from GAMES Magazine, August 1985, p. 30*
Session 14: Fourth Dimension - Meeting Notes

“Future historians of science may well record that one of the greatest conceptual revolutions in the twentieth-century science was the realization that hyperspace may be the key to unlock the deepest secrets of nature and Creation itself.”

Michio Kaku, physicist (b. 1947)

INTRODUCTION AND SUGGESTIONS:

Session 14 would normally be the final session of a semester. Typically we end with a “just for fun” topic. This one is the fourth dimension. My suggestion is to use these materials and to view the hour-long video of Dr. Ed Burger on the fourth dimension for this session. This session can also include a party (e.g. food and beverage) and recognitions of tutors who participated in the competition (if it was held). Another idea is to watch a version of Flatland. Two full-length films were made in 2007 based on the book Flatland (one is better than the other, but I can’t remember which). There are also many, many videos on youtube about the fourth dimension that you may want to consider. The Ed Burger lecture is the best introduction I’ve ever seen on this topic. It is a video of a lecture, so it’s not a slick Hollywood or PBS NOVA presentation, but Dr. Burger has earned awards for his teaching and with good reason. His comedic timing is amazing, and he’s a very engaging speaker.

I would recommend having the students read through the front and back of the first page of the handout and answer the two questions there, then watch the video. I would leave the third page of the handout for them to do on their own later, and I would handout the Problem Set and do an overview with them - pointing out the questions on pages 3 and 4 and allowing for discussion. Resources for further reading are given for the student at the end of the Problem Set.

If there are “back-burner” problems from the semester you may want to consider leaving time during this session, since it is the last of the semester, to allow for discussion on those - or they can just remain as “back-burner” problems that the students can continue to mull over across time - a good lesson in persistence!
“Future historians of science may well record that one of the greatest conceptual revolutions in the twentieth-century science was the realization that hyperspace may be the key to unlock the deepest secrets of nature and Creation itself.”

Michio Kaku, physicist (b. 1947)

One method of trying to gain insight into the fourth dimension is to imagine three-dimensional space from the perspective of a two-dimensional being (a “flatlander” or “Ardean”). Another way to try to gain such insight is to imagine analogs of cubes in various dimensions, a two-dimensional square, a three-dimensional cube, and a four-dimensional hypercube give us a good start. Each of these objects can be created from it’s analog one dimension below by a process of copying the shape and connecting the vertices.

There are a variety of ways in which to represent a cube. Probably the most common way is the ‘Necker’ Cube pictured on the left. Which square is in the front, and which square is in the back? If you look at a Necker Cube long enough you can get it to pop back and forth. Another method is to use perspective, to draw a square within a square, as if one is looking down into the cube. Yet another method is to unfold the cube, as shown on the right.

None of the above is actually a cube, of course; each one is a two-dimensional representation of a cube. Because we are used to these representations we can see them as cubes and understand what is meant, but cubes have depth to them - they have six equal faces - all of their angles are right angles. None of the above images depicts all of these elements accurately.
Similarly to what was done on the previous page, we can represent a hypercube in a ‘Necker-type’ format or as a cube within a cube or as an unfolded object.

1) I imagine you could easily see on the previous page how you would fold the bottom right-hand image into a cube. How would you fold the right-hand image above into a hypercube?

2) We know there are 6 squares in a cube. (That’s why standard dice have 6 sides!) Use the image on the right above to determine how many cubes are in a hypercube.
I hope you discovered on the previous page that a hypercube contains 8 cubes. Below you see multiple copies of two different models of a hypercube. One model corresponds to the Necker Cube, and one corresponds to the square within a square model. Choose one (or both!) models, and try to find each of the 8 cubes. Use a highlighter or crayon to outline each cube as you find it.
Session 14: Fourth Dimension - Problem Set

“Future historians of science may well record that one of the greatest conceptual revolutions in the twentieth-century science was the realization that hyperspace may be the key to unlock the deepest secrets of nature and Creation itself.”

Michio Kaku, physicist (b. 1947)

This is the last seminar of the semester. In our semester-ending seminars I like to cover things that are a little lighter or more whimsical but still thought-provoking. In this seminar we will consider dimensionality, particularly the fourth-dimension.

The idea of a fourth dimension of space was a hot topic in popular culture during the late 1800s and early 1900s. One conception of the fourth dimension is that it is a spiritual realm, the dwelling-place of angels and ghosts. Séances were a popular past-time during that period. That all sounds very non-mathematical and non-scientific, but fourth or higher dimensions are often spoken of in mathematics and science, and some place in which we hear about higher dimensions are in Einstein’s Theory of Relativity and in the physics of String Theory. This can be seen in our quote for this seminar by renowned physicist Michio Kaku.

Our experience is that of three spatial dimensions, and because we are embedded in what we experience to be three-dimensional space it is very difficult to comprehend what an experience of four-dimensional space would be. If we were able to inhabit the fourth dimension we would be able to do all sorts of things that would seem magical or miraculous to those still dwelling in the third dimension. We would be able to perform surgery without cutting the patient’s skin. We would be able to rob a safe without opening it. We would be able to reverse ourselves (for instance we could move into the fourth dimension and come back with our heart in the right side of our chest instead of the left). How can we begin to understand what that kind of reality is like? One approach is to drop down a dimension and try to understand how a two-dimensional being might be able to comprehend a three-dimensional reality. This was done most famously by Edwin Abbott in his book Flatland: A Romance of Many Dimensions, written in 1884.
In Abbott’s book the beings are polygons and lines that live in the plane. Women are lines, men are polygons. The more sides you have the higher you rank in society. Women are the lowest rank, but they are also very dangerous because they can use their sharp ends to puncture the men! Of course their houses, as you can see below, are different from ours:

![Diagram of a house in Abbott's book](image)

Something that isn’t quite a good analogy to our world in Abbott’s book, however, is the fact that the two-dimensions beings have full freedom of movement within the plane. We don’t have full freedom of movement in our three-dimensional world. We are held to our planet by gravity, and though there are ways to overcome that - jumping, getting in an airplane, getting in a rocket - all of those take significant effort. Another approach to considering a two-dimensional world is that of A. K. Dewdney in his 1984 book *Planiverse*. In it he imagines the beings held by gravity to the surface of a circle, as we are held to our spherical earth. What would it be like to exist in the Planiverse? Beings who were left-facing and beings who were right-facing might have to walk all the way around the planet in order to have a face-to-face conversation:

![Diagram of beings walking around a circle](image)

Can you see how such a being would have to be able to access the third dimension in order to turn around?

![Diagram showing how a being might access the third dimension](image)

Feels restrictive and claustrophobic to think of living in a two-dimensional world, but we also are restricted by not being able to access more than our three dimensions!!
What other issues besides whether or not they are able to talk face-to-face might these Dewdney’s two-dimensional creatures have? Consider the following questions:

1. What sort of biology would they have? (If they had a digestive system open on both ends they would fall apart, having been split in half! And would people be left-facing or right-facing as we’ve imagined on the previous page?)

2. Could people build houses on top of their planet? If so, how would people move around?

3. If someone was walking toward you, how could you walk around that person to get past them?

4. What sorts of sports or games might they be able to play? Could such a being juggle?

Here are some images that illustrate how Dewdney and others have dealt with these questions. What do you think is being addressed in each image, and how does it solve a problem that has been posed?
Further questions:
- What other sorts of structures might exist in this world?
- What types of technology might they have?
- How might we be restricted dimensionally in ways you hadn’t thought of before?
- How could you do the ‘magical’ things I mentioned on the first page of this document if you were a four-dimensional being?

---

**FOR FURTHER READING:**

Rudy Rucker’s *The Fourth Dimension: A Guided Tour of the Higher Universes*

Clifford Pickover’s *Surfing through Hyperspace: Understanding Higher Universes in Six Easy Lessons*

Herbert Kohl’s *Mathematical Puzzlements: Play and Invention with Mathematics* (pages 47-55)