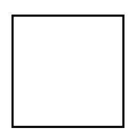
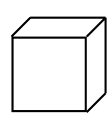
Toward an Understanding of Fractal Dimension

There are multiple ways of defining dimension, and the definition chosen has to do with the context of what it is that is under consideration. Informally dimension is most often understood in one of two ways:

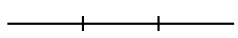
1) as the minimum number of coordinates needed to specify a point
2) as degrees of freedom

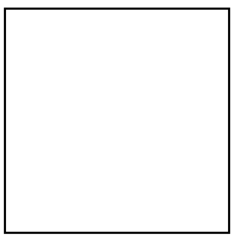
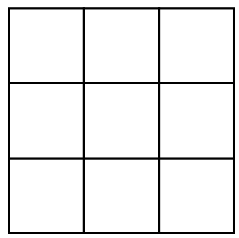
A line segment is considered 1-dimensional because we only need one piece of information to specify a point, and because our freedom of movement is limited to one direction (and the opposite of that direction). A plane is considered to be 2-dimensional because two pieces of information (an 'ordered pair') are required to specify a point, and because our freedom of movement is limited to two directions (up/down and left/right).
The common understanding of dimension is as follows:

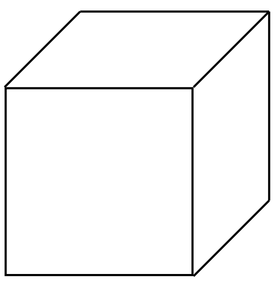
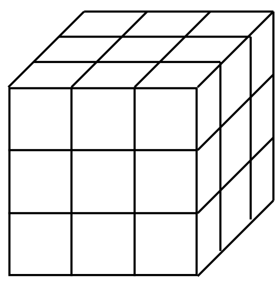
Dimension	Representative Shape	Image
0	point	•
1	line segment	
2	square	
3	cube	

On the next page we will consider what happens when we take these shapes and cut each of their edges into a given number of pieces. We will focus only on the 1, 2, and 3 dimensional shapes, and we will consider equations related to each situation.

Each edge (line segment) of each shape below has been cut into three equal pieces. How many smaller copies of the original shape appear after such cutting? What is the relationship between the number of cuts of each edge to the number of smaller copies of the original?

 $\Rightarrow$  $\Rightarrow$ $3^1 = 3$

 $\Rightarrow$  $\Rightarrow$ $3^2 = 9$

 $\Rightarrow$  $\Rightarrow$ $3^3 = 27$

On the left side of each equation is a 3, representing the number of pieces each edge was cut into. On the right side of each equation is the number of smaller copies of the original shape that are created by these cuts; for example, the square is cut into 9 smaller squares. These numbers have been related through the use of exponents. **NOTICE that in each case the exponent IS the dimension of that shape.** Considering dimension in this way is called *self-similarity dimension* and is one way of finding the dimension of a fractal. Thus the formula for self-similarity dimension is:

$$(\text{\# of pieces each edge was cut into})^{(dimension)} = (\text{\# of smaller copies of the original})$$

Letting E stand for the number of pieces an edge was cut into, letting D stand for dimension, and letting C stand for number of smaller copies of the original, we have our formula more briefly:

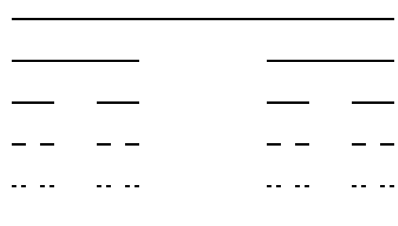
$$E^D = C$$

What are fractals? Why do we need a new way of looking at dimension in order to work with fractals? Why don't the old, informal methods work? Well, let's consider some fractal shapes and observe how they behave.

Fractals are not static shapes like the shapes in a high school geometry class. A fractal is a shape that is the result of an ongoing process of iteration. Classic fractals (the most basic ones we can look at) begin either as a line segment, a triangle or a square. The shape is then cut up and changed in some way (usually by adding additional pieces or by removing pieces). Whatever was done in this first step gets carried out over and over, and, math being math, we can determine what happens if this is carried out infinitely many times.

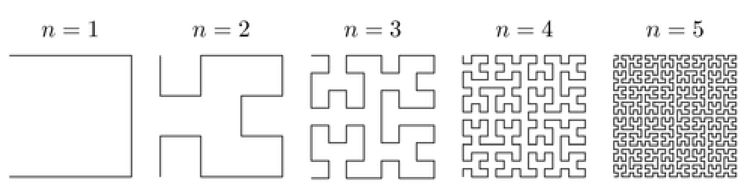
The first fractal pictured on this page is the **Cantor Set**. It is created by beginning with a line segment, cutting it into three equal pieces and then removing the middle third. Once this has been done we now have 2 smaller segments left, and this process is done on each of them. This continues to infinity. Once the process has been completed there is no length left - only a dusting of infinitely many points spread out across where the original segment was. So, the question is, what dimension does the fractal have? It starts out as a one dimensional segment, but all the length gets removed, but not everything goes away - we're still left with infinitely many points, and points are zero dimensional. So, is this one dimensional, zero dimensional, or something in-between (and is that even possible?)

Cantor Set



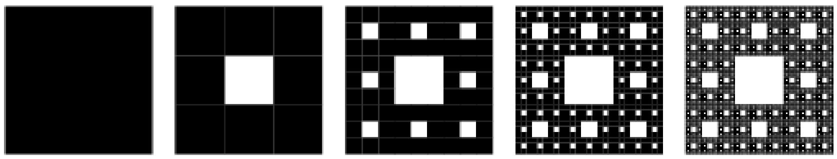
The second curve we'll consider is the **Hilert Curve**. It begins as one or more line segments, upon which a slightly more sophisticated rule than the one above is carried out, and the result is that after infinitely many iterations the curve fills in the entire square - hits every point. Well, a square is two-dimensional, and a line segment is one-dimensional, which means this object because as one-dimensional, but grows to fill a two-dimensional square, so what is the dimension of this shape?

Hilert Curve



Our third and final shape is the **Sierpinski Carpet**. It is created by beginning with a square, cutting it up like a tic-tac-toe board and removing the middle square. This now leaves 8 smaller squares, and this same process is repeated on each of these. After infinitely many iterations (yes, we can do that in math!), all the area goes away, but there is infinite perimeter left. What dimension is this shape that started out as a two-dimensional square but loses all its area, yet leaves infinitely much perimeter?

Sierpinski Carpet



We will look at how to determine the dimension using our previously developed definition of *self-similarity dimension*, but first a question. If the word 'geometry' means, literally, 'earth measure,' how can these shapes be part of 'geometry?' They certainly seem very artificial and not very familiar from the world around us. In our world we see honeycombs made of hexagons. We see circular ripples made in the smooth water of a pond. We see our spherical earth. True . . . but think of trees, bushes, clouds, shorelines, the jagged edge of flame and rising smoke from a campfire, river systems, etc., etc., etc., etc. These jagged shapes are actually much better at describing much of the real world than classical geometry is.

Let's answer the questions about dimension and then look at some applications.

The formula we developed earlier was $E^D = C$. Well, for the **Cantor Set** $E = 3$ (that is the number of pieces an edge was cut into), $C = 2$ (that is the number of copies of the original in the 'second' stage), so we have:

$$3^D = 2$$

Well, that isn't one we recognize the answer to right off, so we're going to have to use algebra to get at that exponent, and this requires logarithms:

$$3^D = 2 \Rightarrow \log 3^D = \log 2 \Rightarrow D \log 3 = \log 2 \Rightarrow D = \frac{\log 2}{\log 3} \approx 0.6309297536$$

In other words, the Cantor Set has a dimension that is in between 0 and 1. This may seem odd, but what this information tells us is how that shape fills up space - an infinite dusting of points across what had been a line segment . . .

We'll also look at the **Sierpinski Carpet**, in which each edge was cut into three parts and in which we kept 8 copies of the original, so we have:

$$3^D = 8$$

Which we solve in the following way:

$$3^D = 8 \Rightarrow \log 3^D = \log 8 \Rightarrow D \log 3 = \log 8 \Rightarrow D = \frac{\log 8}{\log 3} \approx 1.892789261$$

So for the Sierpinski Carpet we find that a shape that began as a 2-dimensional object and became one in which all the area was removed, but which still maintains a square shape and which includes infinite perimeter has a dimension of about 1.89, which is between 2 and 1.

Of what use is this? Much! Applications range from creation of cell-phone antennas to image compression to earlier detection of cancer to determining forged artwork to modeling natural shapes to modeling chaotic processes (such as weather).

One famous application has to do with the nature of coastlines. It really isn't possible to get an accurate measurement of a coastline, because the amount of detail you can measure depends on the size of your measuring stick - miles, kilometers, meters, feet, inches, centimeters, millimeters. The smaller your measuring stick the more detail you can measure, and the longer your measurement becomes, so perhaps a better question than 'What is the length of a coastline?' would be 'What is the relative roughness of this coastline?' Imagine the coastlines of South Africa (very smooth), Portugal, England, and Norway (very rough with all its fjords). Their fractal dimensions are, respectively, 1.05, 1.13, 1.25, and 1.52. This gives us information about the way these 'curves' fill 'space.'

Consider the artwork of Jackson Pollock, also known as 'Jack the Dripper.'

His paintings are worth millions, so it's not surprising that people who think they can copy his work ('just' dripping paint, right?) might try to create and pass off a forgery. There are many tools in art for determining the authenticity of artwork, and one among them (at least for Pollock's work) is to consider its fractal dimension. His work increased in fractal dimension over time, so if the painting is dated it can be compared with others before and after in terms of dimension to see if that test is a fit for authenticity or if it points to it being a forgery.

Shapes such as coastlines and drip paintings are more complicated than the fractals we looked at here, so the method for determining fractal dimension is slightly more complicated, but not much. It is related to our method, which simply comes down to the formula $E^D = C$. Bottom line - there are many ways of defining dimension, and the way you go about it in any given situation is based on what information is needed. **This particular type of dimension give us information about how a shape fills space.**