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INTRODUCTION

WARNING: This class will be unlike any other math class you have ever taken. Most people who have taken this class find that to be a good thing. I hope you will too! But I just need to let you know to enter here with an open mind and, as much as possible, without preconceived notions about mathematics. Also, many people take this class having heard from a counselor or friend that it is an easy math class. Well, it might be “easier” than calculus, depending on your definition of the word “easy.” I would say most people find the class intriguing and different but not easy. It is a college-level transferable mathematics class after all, so please do not enter with any delusions of ease or you will only be doing yourself a disservice. That said, I think you’ll be successful and also really enjoy the class if you come at it with a bit of curiosity, an open mind, and a readiness to work.

For most students taking this course it is a terminal class - not meaning it will kill you but that it is the last math you ever have to take. Before you leave your mathematics career behind it’s important that you know what Math (capital M) really is. It is NOT just crunching numbers and solving for $x$, far from it! It is fractals and time time travel and logic and infinity and beyond! My vision for how I approach this class is well-expressed by two authors that I’ll be paraphrasing/quoting below.

As a math teacher I’m often asked “Why do I have to take math?” or “What is math good for?” This often comes up when things get particularly stressful, such as when algebra students are learning long division on polynomials. I think the answer the person is hoping for is a specific justification of that very specific topic and how exactly he or she will use that one item on the job later in life. Math is certainly very applicable and practical. Without it we wouldn’t have cell phones or airplanes or skyscrapers or television or credit cards or computers or pace-makers or medical imaging or much else. Most people want to see math as a tool box and know how to use each tool inside the box. Really, math is much more than that and much more important than that. Yes it is applicable, but more than that it is a way of thinking and a way of seeing.

Paraphrase from Ian Stewart’s Nature’s Numbers pp. 27-29:

How mathematics earns it’s keep is through practical applications. Our world rests on mathematical foundations, and mathematics is unavoidably embedded in our global culture. The only reason you don’t always realize just how strongly your life is affected by mathematics is that, for sensible reasons, it is kept as far as possible behind the scenes. When you go to the travel agent and book a vacation, you don’t need to understand the intricate mathematical and physical theories that make it possible to design computers and telephone lines, the optimization routines that schedule as many flights as possible around any particularly airport, or the signal-processing methods used to provide accurate radar images for the pilots. When you watch a television program, you don’t need to understand the three-dimensional geometry used to produce special effects on the screen, the coding methods used to transmit TV signals by satellite, the mathematical methods used to solve the equations for the orbital motion of the satellite, the thousands of different applications of mathematics during every step of the manufacture of every component of the spacecraft that launched the satellite into position. When a farmer plants a new strain of potatoes, he does not need to know the statistical theories of genetics that identified which genes made that particular type of plant resistant to disease.

But somebody had to understand all these things in the past, otherwise airliners, televisions, spacecraft, and disease-resistant potatoes wouldn’t have been invented. And somebody has to understand all these things now too, otherwise they won’t continue to function. And somebody has to be inventing the new mathematics in the future, able to solve problems that either have not arisen before or have hitherto proved intractable, otherwise our society will fall apart when change requires solutions to new problems or new solutions to old problems. If mathematics, including everything that rests on it, were somehow suddenly to be withdrawn from our world, human society would collapse in an instant. And if mathematics were to be frozen, so that it never went a single step farther, our civilization would start to go backward.

Mathematics is not just a research-driven endeavor that is developed with a specific goal in mind. Often mathematical play out of curiosity gives rise to a profound and unexpected application. In fact, one of the
strangest features of the relationship between mathematics and the “real world,” but also the strongest, is that good mathematics, whatever its source, even if it came from mere curious play, eventually turns out to be useful. For instance, the mathematical study of knot theory originated when scientists thought atoms might be best described as knotted vortices in luminiferous ether. Well, it turned out that atoms are nothing like knots, but mathematicians got curious and kept playing with knots even though they had no application at all. Decades later, once chemistry, biology and physics had progressed - for instance once DNA had been discovered - knot theory was found to have many applications including to the workings of DNA. In the 1600s there was an interest in the vibration of violin strings. Three hundred years later that led to the discovery of radio waves and the invention of radio, radar, and television.

Curiosity in mathematics, playing with things that may or may not seem to have an application, often ends up resulting in important breakthroughs that were entirely unpredictable. There is a dance in mathematics between play and application and both are equally important.

Paraphrase from Arthur Michelson’s article Why Math Always Counts (LA Times, December 26, 2004)

Math is not just about computing quadratic equations, knowing geometric proofs or balancing a checkbook. And it’s not just about training Americans to become scientists. It has implicit value. [Math] is about discipline, precision, thoroughness and meticulous analysis. It helps you see patterns, develops your logic skills, teaches you to concentrate and to separate truth from falsehood. These are abilities and qualities that distinguish successful people. Math helps you make wise financial decisions, but also informs you so you can avoid false claims from advertisers, politicians and others. It helps you determine risk. Some examples:

** If a fair coin is tossed and eight heads come up in a row, most adults would gamble that the next toss would come up tails. But a coin has no memory. There is always a 50-50 chance. See you at the casino?

** If you have no sense of big numbers, you can’t evaluate the consequences of how government spends your money. Why should we worry? Let our kids deal with it

** Enormous amounts of money are spent on quack medicine. Many people will reject sound scientific studies on drugs or nutrition if the results don’t fit their preconceived notions, yet they might leap to action after reading news stories on the results of small, inconclusive or poorly run studies.

** After an airplane crash, studies show that people are more likely to drive than take a plane despite the fact that they are much more likely to be killed or injured while driving. Planes are not like copycat criminals. A plane is not more likely to crash just because another recently did. In fact, the most dangerous time to drive is probably right after a plane crash because so many more people are on the road.

The precision of math, like poetry, gets to the heart of things. It can increase our awareness. It is not possible to really understand science and the scientific method without understanding math. A rainbow is even more beautiful and amazing when we understand it. So is a lightning bolt, an ant or ourselves. Math gives us a powerful tool to understand our universe. I don’t wish to overstate: Poetry, music, literature and the fine and performing arts are also gateways to beauty. Nothing we study is a waste. But the precision of math helps refine how we think in a very special way.
MA THEMA TICIAN PRESENT A TION

INTRODUCTION Of the six units we will cover in class the fifth one is a history of math unit. The way we will cover this unit is that each student will research one mathematician and give a presentation about that mathematician’s life and work. This presentation will be worth 100 points.

GOAL You are to share about your mathematician in such a way that your classmates get a clear, accurate and memorable presentation of his or her life and work. The math history section will be on the final exam, so do a good job of teaching your classmates well!

GETTING STARTED First you will need to select a mathematician. You'll want to get started with this right away so that you make sure you find someone that will work well for you - someone whose life or work you’re interested in - or someone whose life has a lot of drama to talk about or whose work is easy for you to explain. You need to choose one of the mathematicians from the list provided on page 8 here. Resources for making your decision include links on my website and books at the MJC library, as well as Google searches, of course. Each mathematician may be chosen by only one student per class, so choice is on a “first come, first served” basis. Once you make a selection email your choice to me. Your selection is due on or before first meeting of the third week of class. If you have not emailed me a choice by then, I will send around a sign-up sheet that day so that you can choose one of the remaining mathematicians.

REQUIREMENTS Your presentation must involve the computer. Powerpoint, Prezi, or other such programs are excellent choices. If you are not sure how to use one of these programs, seek help (early) at a computer lab on campus or in my office. In addition to a computer-based presentation you may show work on the board as well if you would like to do so. Your presentation should be between 5 and 10 minutes long. Fewer than 5 minutes isn’t long enough to get enough information across, but more than 10 minutes will push into someone else’s time. We only have 3 days for about 40 presentations, so you need to stay within the time frame. Use appropriate resources (i.e. books and/or journals as well as trustworthy internet sites - you may use Wikipedia in addition to other resources, but that cannot be your only resource or your main resource!). You need to present both about your mathematician’s life and work, and part of that needs to be a clear explanation of one piece of the person’s mathematical work. See below for details of due dates and scoring.

DATES

Choice of Mathematician   DUE: Week 3   Meeting 1
Questions 1-3 (page 9)     DUE: Week 6   Meeting 2
Outline of presentation (page 10)  DUE: Week 10  Meeting 2
Presentations             BEGIN: Week 12  Meeting 1 (continuing until finished)

Note: EVERYONE should be ready to present on the first day!!

SCORING This project is worth a total of 100 points, which is equivalent to the value of a test. The points are earned (or lost) as follows:

1. 30 points: following directions and meeting the deadlines for turning in topics and outline (@ 15 pts.)
2. 10 points: using a computer program such as power-point or prezi effectively to present your work
3. 10 points: speaking slowly enough, clearly enough, and loudly enough to be heard
4. 25 points: sharing interesting and accurate biographical data in an engaging manner
5. 25 points: clearly and correctly explaining at least one aspect of the person’s mathematical work
6. deductions taken for going overtime or under-time, certain mispronunciations, negativity, avoidable technological glitches, or being absent on your day to present.
POINTS & DEDUCTIONS This should be an easy A for you if you simply follow what is asked and make a good-faith effort. Keep in mind that this is worth 100 points, which is equivalent to a test, and this is totally in your control. Also keep in mind that as you work on it you have librarians who are happy to help you find good resources, and you may come in and ask me questions as you put this together (as long as you do so before week 11 of the semester - as I do not want to facilitate last-minute work). If you simply fill in pages 9 and 10 in this document and turn them in properly filled out and on time you earn 30 points, which is equal to 3 letter grades on a test! Take advantage of all this!! My hope is that everyone will get 100%. So I also include a heads up about deductions. If you do even not bother to find out how to pronounce the name of your mathematician, points will be deducted. I will also take off points for mispronunciations of significant places in your mathematician’s life (city of birth, university taught at, etc.). If you are not sure about a pronunciation, just come ask me! Easy enough! Deductions will be taken for each minute (or part of a minute) below or above the 5 to 10 minute requirement. Also, I have noticed in recent years that when some people get nervous giving a presentation they say self-deprecating things, such as, “Sorry this is so stupid” or “Sorry this is so boring” or “Well, I didn’t really study this, so I don’t know what I’m talking about.” You’ve got 11 weeks to prepare - don’t let it be boring, and if there’s something you don’t understand, come in and ask me! If you still don’t feel good about your presentation on the day you give it, just put on a smile and fake it!

CORRECT PRONUNCIATIONS OF COMMONLY MISPronOUNCED WORDS:

- Euler = OY-ler (not YOU-ler)
- Erdős = air-dish (not err-dose)
- Srinivasa = shree-nee-vah-sah
- Cambridge = came-bridge (not camm-bridge)
- Königsberg = con-igs-berg is acceptable (though with the umlaut it’s really more like kerr-nigs-berg)
- Ecole Polytechnique = ay-cole poe-lee-teck-neck
- Göttingen = guh-ting-in
- Vassar = vá-sir (not v-såhr)
- treatise = trea-tiss (not trea-tize or trea-tease)

RESOURCE SUGGESTIONS (a few of many!!)

- Your textbook for this class has lots of short bios!
- Mathematicians are People Too by L & R Reimer (in MJC library)
- Mathematical People by D. Albers (in MJC library)
- A History of Mathematics by C. Boyer (in MJC library)
- Great Feuds in Mathematics: Ten of the Liveliest Disputes Ever by H. Hellman (in MJC library)
- The Greatest Mathematicians of All Time trustworthy website linked on our class page
- MacTutor History of Mathematics: Indexes of Biographies trustworthy website linked on our class page

Most of the above are compilations; look also for individual biographies. For instance My Brain is Open and The Man Who Loved Only Numbers are great bios of Paul Erdős. An excellent (and short!) new biography out of Fibonacci is The Man of Numbers. For Galois there is the classic biography Whom the Gods Love (by Infeld), and for Lewis Carroll (yes, he was a mathematician!), there is the recent Lewis Carroll in Numberland: His Fantastical Mathematical Logical Life. Be careful about using feature movies about mathematicians - example A Beautiful Mind, which is about John Nash; it’s a great movie, but it is very fictionalized! However, there is a PBS presentation about that movie and his life that you can probably find online; it’s called A Brilliant Madness. There are also NOVA presentations about Ramanujan and Wiles.
OTHER INFO - HINTS, TIPS, POINTERS AND WARNINGS

• Because we have 40 presentations and 3 days, timing is an issue. Practice at home to make sure you are within the 5 to 10 minute limit. (Points will be deducted if you are outside those boundaries.)

• Your time up front is short. Focus on interesting parts of the mathematician’s life and on the one piece of mathematics you’ll be sharing with us. DO NOT list 5 schools he attended and 10 books she wrote and 15 awards he received and the 7 schools she taught at - imagine hearing 40 such lists in a row - not memorable stuff!!

• Be alert to presentation common sense. For instance, don’t use small font or put so many words on one slide that the class can’t read them. Because your classmates will be tested on this, they will be stressed out if they see 100 words on a slide, and they’re trying to write them all down. Use slides to highlight important information but not as a way to write out every word you want to say. Do include some pictures and/or equations and/or diagrams and/or maps to help the class understand your mathematician’s life and work.

• Technology issue #1 - technology can be ‘glitchy,’ so be sure you have multiple ways to access your presentation. For instance email your presentation to yourself AND bring it in on a flashdrive. Sometimes the internet doesn’t work; sometimes a computer decides not to like your drive, so have at least 2 methods of accessing your presentation. If you do not do so and your presentation won’t load, you will lose points. (Note that you should email your presentation to yourself, not to me. If I have 80 power-point presentations sent to my inbox - well - yeah - that’s just not going to work.)

• Technology issue #2 - we have PCs in the classroom. If you are working on a Mac, be sure to save your presentation in such a way that it will play on a PC (or use a computer in an MJC lab to create your presentation). Incorrect formatting is also an avoidable mistake, and points will be taken off if your presentation won’t run due to a choice that was within your control such as this.

• Yes, you are expected to EXPLAIN some math. That doesn’t mean you’re going to give a full blown math lecture. Choose a small piece of the person’s work, or, if you choose someone like Sir Isaac Newton, who is a father of calculus, go ahead and give us an overview of what calculus is about; you’ll be surprised how easy an overview like that is, and I’m happy to help (as you talk to me before week 11)! Don’t just say, “He invented Venn Diagrams” or “He’s the Father of Calculus” or “We did Cramer’s Rule in Math 90, so I know you already know it.” Help us understand these things and why they’re important or what they do or where they came from or why.

• You are welcome to do pretty much (within reason) whatever makes you comfortable presenting. You may sit down or stand up. You are more than welcome to use note-cards. You can give the class a handout (if you think it will help them and take pressure and focus off of you). This isn’t a speech class. I won’t be grading on eye-contact and posture and such things, but one way or another you need to get the information across clearly.

• Realize that prior to the 1500s or so not much information was kept about the lives of individual people, unless they were royalty or VERY famous or something, and sometimes not even then! If you choose a mathematician from ancient times and can’t find much information about that person’s life, tell us whatever little bit is known or guessed about his or her life, and also just tell us about life and culture at that time, which is also a way of telling us about that person’s life.

• Some of the more recent mathematicians can be found on YouTube! If you find a clip of your mathematician you’re welcome to include it, but it shouldn’t be more than 1 minute long.

• If your mathematician had a major conflict with another one on our list (such as Newton and Leibniz or Cantor and Kronecker or Tartaglia and Cardano), be sure to at least mention that and maybe even really get into that! One book in the MJC library is all about these famous mathematical feuds!!

REASSURANCE I hope all the information in these pages hasn’t overwhelmed you or stressed you out. I’m just trying to get you all the information (and advice) you need. Most students get 100% on this or nearly 100%, and I’m sure you can too if you simply take it seriously and don’t put it off to the last minute.
MATHEMATICIANS TO CHOOSE FROM

Your choice of mathematician does need to come from this list. There is a lot of variety here going from ancient times to the present day - including men and women - including a variety of mathematical specialties - and including a variety of nationalities. If you look closely you may recognize some names - especially those of a couple of famous writers and a current actress who also happens to be a mathematician! Remember that choices are ‘first come, first served,’ so as soon as you make a selection email me your choice. **In your email be sure to include your name, the name of your mathematician, and either your class section number or the time your class starts.** I need to know who you are and what class you are in or I cannot properly record this!

<table>
<thead>
<tr>
<th>Mathematician</th>
<th>Mathematician</th>
<th>Mathematician</th>
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<tbody>
<tr>
<td>Thales</td>
<td>Pythagoras</td>
<td>Zeno of Elea</td>
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<tr>
<td>Euclid</td>
<td>Archimedes</td>
<td>Eratosthenes</td>
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<tr>
<td>Diophantus</td>
<td>Hypatia</td>
<td>Omar Khayyam</td>
</tr>
<tr>
<td>Fibonacci</td>
<td>Nicolo Tartaglia</td>
<td>Girolamo Cardano</td>
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<tr>
<td>John Napier</td>
<td>Rene Descartes</td>
<td>Blaise Pascal</td>
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<tr>
<td>Isaac Newton</td>
<td>Gottfried W. v. Leibniz</td>
<td>Gabriel Cramer</td>
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<tr>
<td>Leonhard Euler</td>
<td>Maria Aguesi</td>
<td>Sophie Germain</td>
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<tr>
<td>Carl F. Gauss</td>
<td>Nikolai Lobachevsky</td>
<td>Carl Jacobi</td>
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<tr>
<td>W. R. Hamilton</td>
<td>George Boole</td>
<td>Leopold Kronecker</td>
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<tr>
<td>Bernhard Riemann</td>
<td>Lewis Carroll</td>
<td>John Venn</td>
</tr>
<tr>
<td>Georg Cantor</td>
<td>David Hilbert</td>
<td>Srinivasa Ramanujan</td>
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<tr>
<td>Gaston Julia</td>
<td>John von Neumann</td>
<td>Piet Hein</td>
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<tr>
<td>Kurt Gödel</td>
<td>Grace Hopper</td>
<td>Paul Erdős</td>
</tr>
<tr>
<td>Benoit Mandelbrot</td>
<td>John Nash</td>
<td>Roger Penrose</td>
</tr>
<tr>
<td>Edward Thorp</td>
<td>Andrew Wiles</td>
<td>Danica McKellar</td>
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</tbody>
</table>

**COMMENT** For the list I have provided here I have chosen mathematicians who either had very interesting lives, did math that is reasonable for you to look into, and/or were extremely major figures in mathematics. Some of their lives are very ‘interesting’, by which I mean quirky and eccentric perhaps to the point of madness. This does not mean most mathematicians are crazy! I just tried to provide choices for you that would be especially interesting and memorable!

**MOST IMPORTANT** For the 5 or 10 minutes you are up front, you are the teacher. Your mathematician is real person who lived (or is living) a real life. During the semester, as you prepare, look at it as if you are adopting him or her and then telling your classmates about a friend you care about!
DIRECTIONS: Be sure to read all requirements for the mathematician presentation in your Extended Syllabus. You will turn in this paper twice, once during week 6 with the three questions on this page answered, and once during week 10 with the back side filled out.

In recent years I’ve gotten responses to the questions below that are unrelated to what I’m asking. DON’T MAKE THAT MISTAKE! READ THE FOLLOWING SENTENCES CAREFULLY BEFORE RESPONDING IN ORDER TO BE SURE YOU ARE RESPONDING ACCURATELY.

1) In a few brief sentences express two or three interesting stories, events or anecdotes you will be sharing about your mathematician’s LIFE aside from his or her work (i.e. DON’T write in this space about his or her mathematical work or teaching or education).

2) In the space below use one full sentence to tell me the ONE specific major mathematical contribution of your mathematician that you will be explaining to the class. Be sure to choose a mathematical topic this person worked with that you can explain! (You can tell the class about more than one during your talk if you want to, but I just want to know about one right now.)

3) Put the YEAR of your mathematician’s birth in the blank at right: __________

************************************************************************************

SCORING - THIS SECTION FOR INSTRUCTOR USE ONLY

- Top front of this page filled out CORRECTLY by due date: 0 __________ 15
- Back of this page filled out by due date: 0 __________ 15
- Presentation - powerpoint or prezi: 0 __________ 10
- Presentation - audibility: 0 __________ 10
- Presentation - biographical content: 0 __________ 25
- Presentation - mathematical content: 0 __________ 25
- Point deductions (see directions) minus __________

TOTAL: __________
DIRECTIONS: Write out an *outline* of your presentation (using bullet points only - not full paragraphs). I am looking for all the topics you plan to present in the order you plan to present them. Write this out below on the *left side ONLY*. I will be using the right side to take notes and score your presentation as you speak. MAKE A COPY for yourself; this will *not* be returned before presentations, though you are welcome to come into my office and go over it with me in person if you’d like.
WHAT IS MATH?

Begin working on this as I am taking attendance and doing add cards. We will go over it later in class today and also at our next class meeting. When we go over it together I will be using it to express my view of mathematics, a view that I hope will be helpful to you. Once this is returned, keep it in your notebook throughout the semester as a reminder.

1) Without lifting your pencil from the paper and without retracing any of your lines, connect all 9 dots below using exactly 4 straight lines.

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  .  .  .  
  .  .  .  
  .  .  .  
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2) Determine the common word or phrase represented by each box below. For example, the first box represents the phrase “reading between the lines.”

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<table>
<thead>
<tr>
<th>reading</th>
<th>TOUCH</th>
<th>STAND</th>
<th>MAN BOARD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>0 M.D. Ph.D. BA</td>
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</tbody>
</table>
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3) How many SQUARES of any size are there in this diagram? The answer is not 25, and it is not 26. It is larger than either of those numbers. Remember the SQUARES can be of any size, and your goal is to find and count all of them. (Recall that a square is a four-sided figure that has the same height and width.)

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The following figures are all graphs. A graph is called traversable if all of the edges can be traced without lifting your pen from the paper and without going over any edge more than one time. Determine which of the following graphs are traversable and record your information on the next page. **NOTE**: Some graphs that are traversable may not seem so at first. For instance, with figure 2 if you being at the lower left-hand vertex, you cannot traverse the graph, but this does not mean it is not traversable. If you begin at the upper left-hand vertex, you can easily traverse this graph. Before stating a graph is not traversable, make many attempts starting at different points, taking different turning until you are fully convinced it is not traversable. Compare with others around you to see if they agree.
On the previous page you were asked to determine if each of the graphs pictured was traversable. Record your findings on this sheet in column 2 of the chart. Then go back to handout one and determine for each graph how many even vertices it contains and how many odd vertices it contains, and record that information here as well. Then use the chart to look for a pattern in order to determine in a way other than trial and error whether a graph is traversable or not. NOTE: A vertex is odd if it has an odd number of edges connecting to it, and it is even if it even if it has an even number of edges connecting to it.

<table>
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<tr>
<th>GRAPH</th>
<th>TRAVERSABLE? (Yes or No)</th>
<th># of ODD vertices</th>
<th># of EVEN vertices</th>
<th>Does is matter where you start?</th>
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<td>figure 1</td>
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With the graphs you were working with on Graph Handout 1, you were using trial and error to decide if a graph was traversable or not. But there are problems with this method; not only is it tedious for complicated graphs, but unless you can be sure you’ve tried every possibility you can never be confident in saying a graph is not traversable.

Use the chart above to look for a pattern. Find a pattern that you can use to tell you whether a graph is traversable or not based only on this information and not on having to do trial and error.

Now look again at the pattern, if a graph is traversable, how can you tell whether or not where you start matters? And, if it does matter where you start, how can you tell where to start?

CHECK to see if your answers above work! Create your own graphs similar to the ones you’ve been working with; create different configurations of them with different numbers of even and odd vertices and see if your answers work. If so, congratulations! If not, rethink your answers and tweak them as needed.
1) The following diagram is of a 5-room house. In the large blank space at the bottom of this page, draw a graph consisting of vertices and edges to model this diagram. Your vertices should represent locations, and your edges should represent connections. Use your graph to determine if it is possible to visit every room using each door exactly once. If it is possible, draw this path on the “house” to show the answer. If it is not possible, explain why in a mathematically precise way.
2) Here are two additional “houses.” The question you’re being asked regarding each of these is the same as in problem #1 on the previous page. Be sure to address each part of the directions.

\[a\) \quad b\]

3) Below is a situation similar to that of the Konigsberg Bridge Problem that was discussed in class and in your textbook. Use what you know of graph theory to determine if an Eulerian trail or Eulerian circuit exists here. Explain your answer using one or two full sentences using the ideas of graph theory.
4) In class we created a graph that gave all the moves (and solutions) for a 2-disk *Towers of Hanoi* puzzle. Use the space below to create a **GRAPH** for 3-disk *Towers of Hanoi*. It is a graph that is being asked for here, *not* pictures of various stages of solving the puzzle - although you may want to draw pictures of the steps on scratch paper to help you with your work.
5) There is a legend that goes with the *Towers of Hanoi* puzzle, and that is that in an Asian temple monks have the task of solving this puzzle with 64 disks. They work at it in shifts, working around the clock and moving one disk per second; they never make a mistake in their moves. The legend is that when they have completed the task the world will end. How long will it be from the time they began until the time they finish? (A java applet that you can actually play of this puzzle is at the link below.)

http://www.mazeworks.com/hanoi/index.htm
Imagine the vertices in each of the three situations below (triangular, square and pentagonal) to represent objects that need to be connected by wires. In each case, what is the least amount of wire needed to accomplish this? Use a ruler with metric measure so that you can determine this in millimeters for greatest accuracy. Try different configurations of wiring, measure each, and record the length in millimeters. Some questions to think about are the following. Does there need to be a path or circuit involved in order to connect these items and have electricity flowing to each? Might we be able to “patch” the wires together somewhere to help us use less wire? (If so, this would basically mean adding another vertex. Would that help shorten the total distance?)
Instant Insanity is a puzzle that was first marketed by Parker Brothers in 1967. It consists of four cubes, with each face painted one of four different colors (we’ll be using red, blue, yellow and green). The object of the puzzle is to stack the four cubes one on top of the other, so that on each side of the stack each cube face is showing a different color - in other words so that each of the four colors shows on each side.

QUESTION: In general, how easy would this be so solve by trial and error? In other words, how many different ways can you arrange the cubes while stacking like this?

The squares below are provided as a place for you to put your cubes so that you can keep track of them and not lose your place.

Draw a descriptive graph below for each of the cubes above.
Create a composite graph below containing all four graphs from the last page. Label each edge according to the cube it originally came from (A, B, C, or D) - or, rather than labeling, use 4 colored pencils to record which edge came from which cube.

Now, use the edges from the complete graph above to make two sub-graphs below. Each sub-graph should have one edge from each cube (for a total of 4), and each vertex should have order 2 (which will mean that each color is used twice - once each front and back, once each top and bottom).

Use the information from the subgraphs above to write up a chart describing the solution. Make sure that each column has each color listed once only. Rows can have repeated colors.

<table>
<thead>
<tr>
<th>CUBE</th>
<th>FRONT</th>
<th>BACK</th>
<th>TOP</th>
<th>BOTTOM</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
6) Using the formulas we developed in class, find the *angle sum* and the *individual angle measure* for each polygon listed in the table below. Note that we are assuming all of these polygons to be regular. You may be able to simply copy much of this work from your notes, but also determine and fill in the other values that we did not cover.

<table>
<thead>
<tr>
<th>SHAPE</th>
<th>POLYGON ANGLE SUM</th>
<th>INDIVIDUAL ANGLE MEASURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>TRIANGLE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>QUADRILATERAL</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PENTAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HEXAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SEPTAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>OCTAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NONAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DECAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UNDECAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DODECAGON</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Space for scratch work:
7) Extend the work we did in class with angle measure to find the measure of an angle at the tip of a regular pentagram (as noted with an arrow). **HINT: The answer is NOT 60°.** Notice that it is the pentagon that is regular, but this doesn’t mean necessarily that the triangle is!!

![Pentagram Diagram](image)

8) Given the *stellated heptagon* below find the measure of one point of the star. Note that the center is a *regular heptagon*.

![Heptagon Diagram](image)
9) Using what we did in finding angle measures of polygons, your chart on page 19, and our in-class work on determining which polygons tile (tessellate), fill in the chart below:

<table>
<thead>
<tr>
<th>SHAPE</th>
<th>INDIVIDUAL ANGLE MEASURE</th>
<th>WILL IT “TILE” (yes/no)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TRIANGLE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>QUADRILATERAL</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PENTAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HEXAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SEPTAGON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>OCTAGON</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Our in-class explorations with polygons and angle-measure allows us to explore a concept that is both aesthetically pleasing and used in art and is also very practical and used in construction and home improvement. This concept is called tessellation, which means tiling. To tessellate is to tile a surface in such a way that there are no gaps or overlaps. Dutch artist M. C. Escher is known for his tessellations. As part of your assignment you will be asked to create your own tessellation. Below are examples students have created in my previous classes, and on the following pages are examples of M. C. Escher’s work.
Not only did Escher create works that were strict tilings, but he also incorporated these tilings into other images. See figures below. As complex and amazing as they are, all of Escher’s tessellations have a basic geometric shape as their foundation, a polygon that could be used on its own for tiling. These shapes were then altered to become more interesting shapes, such as birds or horses or lizards, but in such a way that they would still tessellate. What shapes can be used as a basis for this? Think about tiling you’ve seen in daily life rather than in art, tiling you’ve seen in floors and counter-tops. These same shapes that can be used in such a basic way as tiling a counter can be altered to create something as fanciful as these works of art.
Here is a description of how a regular polygon that is able to be tessellated can be transformed into another shape that also tessellates. Notice how a piece was cut out of the middle of side AC and taped to the middle of side AB to make a head – and also that parts of side AB were cut and then taped to the edges of side AC to make longer ‘wings’ or ‘fins.’ Then part was cut off of side CB to create a tail and another ‘wing’ or ‘fin.’

Figure 2: Sample Directions for a Tessellation Based on a Triangle

10) On the next page create your own tessellation in the box provided (or create a tessellation on a separate page and paper-clip it to the next page when you turn in your homework journal). HINT: Cut a template out of cardboard or stiff paper to trace around over and over.
11) A picture is shown above of a tessellation that includes more than one type of regular polygon. This sort of tiling is called a *semi-regular tiling*. Using this picture and the angle-measure information you have filled in in the previous two charts, list as many combinations of regular polygons as you can that result in such a semi-regular tiling. Draw pictures of these tilings in the space at the bottom of the page, and fill in the chart below. The first line has been filled in for you as an example. It represents the information about the tiling you see at the top of this page.

<table>
<thead>
<tr>
<th>NUMBER AND TYPE OF POLYGONS USED</th>
<th>SUM OF ANGLES AT EACH INTERSECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 octagons and 1 square</td>
<td>$2 \times 135 + 90 = 270 + 90 = 360$</td>
</tr>
</tbody>
</table>
GEOMETRY (THE FOURTH DIMENSION)

Read over these questions before watching the film on the fourth dimension. As you watch the video of Dr. Edward Burger of Williams College explaining the fourth dimension, keep the following questions in mind, and fill in the answers as you hear them - or after the video if necessary. If you wish to watch the clip again, it is at: http://www.baylor.edu/player/index.php?id=100412&gallery_id=4627

What do mathematicians do when they run into a hard problem?

What benefits would there be medically to being able to access the fourth dimension?

List two other things you could do in the fourth dimension that are not possible in the third dimension.

In the works of art that Dr. Burger shares at the end of his lecture what ideas have painters been able to represent using the fourth dimension?

Dr. Burger explains the fourth dimension, but he says this isn’t really what his talk is about. What does he say his talk is actually about?
FOURTH DIMENSION WRITTEN SUPPLEMENT

As we saw in class the fourth dimension comes in two 'flavors,' that of time and that of space. This supplement will focus on the fourth dimension of space. This can be a hard concept for our minds to comprehend because we only experience 3 spatial dimensions or directions: up/down, left/right, forward/backward. Any other directions we experience are just combinations of these. By its very nature (and name!) the fourth dimension is something that is beyond the three dimensions with which we are surrounded all our lives.

Even though we don’t experience a fourth dimension of space, we can learn about it and explore it by using mathematics and logical reasoning. For the dimensions with which we are familiar there is a simple way to build up from one to the next, and we can use this method to begin to think about the fourth dimension. To build a zero-dimensional, 0D, point into a one-dimensional, 1D, segment, we push the point a distance away from itself and trace out a segment:

\[ \text{OD} \rightarrow \text{1D} \]

Thus, through movement and tracing we have created a one-dimensional object from a zero-dimensional object. Continuing, we push the segment in a direction perpendicular to itself and trace out that movement to create a two-dimensional square:

\[ \text{1D} \rightarrow \text{2D} \]

Continuing in this way we can move from two dimensions to three dimensions. This time we push the square in a direction perpendicular to itself and trace out its path:

\[ \text{2D} \rightarrow \text{3D} \]

Our eyes are trained to see the image on the right above as a cube, but, of course, it is really not a cube. It is a flat drawing on a flat piece of paper (or on a flat computer screen). To create a real cube out of a square we would have to push it up out of the screen, but we cannot do that so we represent that direction using diagonal lines. It’s something we’ve seen enough that we’ve come to accept it.

So in going from a line segment to a square we went in a perpendicular direction (at 90° angles) to what was there, and in going from a square to a cube we went in a perpendicular direction to what we had already (even though the lines we used to trace look diagonal rather than perpendicular, we know the image is supposed to pop off the page, perpendicular to the other lines). We’re up to three dimensions. In order to create a four-dimensional cube, also known as a hypercube, we need to push our cube in a direction perpendicular to all the others we have used so far.

Where is that direction?

It is a direction we cannot see or physically find, just as a 2-dimensional creature living in a world that is a flat plane would have no idea where the third dimension is and would not be able to physically create a
cube. However, we can use the process of tracing we’ve been using all along in order to create a diagram of four-dimensional cube. To do so, we ‘push’ our cube in a direction perpendicular to the others; though we cannot physically do this we can represent it with diagonal lines as we did in going from square to cube.

The cube is already a bit of a stretch, but since we are familiar with cubes in real life we can immediately recognize the representation of one on a flat surface. The 4D cube, or hypercube, on the other hand, is not something we are familiar with, so, though this drawing helps a bit in getting the idea, we’re pretty far removed from anything we can truly visualize.

There are a couple of other ways of approaching this that also involve building up from a cube. Another way to represent a cube is to imagine looking into it as if it were a box:

You know from your experience with cubes and boxes that the square that seems to be in the middle (or on the ‘inside’) here isn’t really on the inside, nor is it really smaller than the other square. It’s just a matter of perspective. If this cube is a box you are looking into, then the bottom is farther away than the top, and therefore appears smaller. Also, you know that a cube has 6 square sides, but this shape appears to only contain two squares. The four other shapes seem to be trapezoids with slanted sides but this is an illusion due to perspective as well. All of the closed figures in this cube actually represent perfect squares. We can take a similar approach to a hypercube. Here is the result:

Here too it looks like we have a smaller shape inside a larger shape and as if we have slanted shapes around the inside, but this is not the case. The fact that one cube looks smaller is again a matter of perspective. It
is simply further away. The shapes with slanted lines around the ‘inside’ are actually cubes. This image is of the same 4D shape as you see at the top of the page but just from a different perspective.

Here is one other option in terms of thinking about the construction of a hypercube based on the construction of a cube. If we wanted to make a paper cube by cutting a model out of a piece of paper and taping sides together, we could use the image on the left in order to accomplish this:

If we wanted to make the model cube, we would just cut out the figure on the left, and fold the sides ‘up’ off the flat plane of the paper and tape them together. This makes perfect sense to us, but to a creature living in 2 dimensions it would be impossible to do and even pretty hard to imagine because he would have no concept of the direction (or dimension) ‘up’ off of the plane. In terms of making a genuine model of a 4-dimensional cube, we find ourselves in the same situation as the flatland creature. We could make our hypercube IF we could figure out where the direction is in which we have to fold it! But we live in the 3rd dimension, so we don’t have enough space in our space, so to speak.

12) Although we cannot “fit” a 4-dimensional (or 5-dimensional) object into our 3-dimensional world, we can use logic and mathematical patterns to discover what higher dimensional objects are like. Just as a cube is a 3-dimensional analog of a square, a hypercube is a 4-dimensional analog of a cube, and a hyper-hypercube is a 5-dimensional analog of a cube. Use logic based on the lecture, the video, and these materials, and use the pattern below to determine how many vertices (i.e. corners), edges (i.e. line segments), faces, cubes and hypercubes are in each of the objects whose rows have been left blank for you to fill in.

<table>
<thead>
<tr>
<th></th>
<th># of vertices</th>
<th># of edges</th>
<th># of faces</th>
<th># of cubes</th>
<th># of hypercubes</th>
<th># of hyper-hypercubes</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>segment</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>square</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>cube</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>hypercube</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>hyper-hypercube</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

13) Time is sometimes looked at as a fourth dimension. Dimensions are sometimes called ‘degrees of freedom.’ We do have freedom of movement in our 3 spatial dimensions: left/right, forward/back, up/down. If time is a dimension like these others it seems like we should be able to move freely in time as well – that is, it seems we should be able to travel in time. For this problem answer parts (a) and (b) on the next page.
(a) Do you think time travel will ever be possible? Why or why not?

(b) Imagine the possibilities of time travel. If you were given one opportunity to travel in time, would you travel to the past or the future? Why? What would you do?
Fractal Geometry involves dynamic processes on shapes - very unlike the way you interact with shapes in a standard high school geometry class. One way to generate fractals is to begin with a standard geometrical shape and to make a change that you then replicate over and over at smaller and smaller scales. The initial shape is called an *initiator* and is considered to be *stage 0*. The change is represented in the next stage, *stage 1*, and is called the *generator*.

In order to understand and work well with fractals you need to develop a vision for what is being done at each stage and for how the sizes of the component shapes of each stage (perimeter and area) are related to the size of the figure in the stage 0, your initial figure. This involves a great deal of work with fractions and just having a sense of fractions.

In the image above you see the initiator and the generator. The initiator, the original large square, was cut twice on each side - making it actually a $3 \times 3$ grid. Then the middle square of that grid was removed.

Q1: How many solid squares are there in stage 1? 

Q2: In comparison to the original square, what size are the squares in stage 1? 

Q3: Compared to the original, what length are the edges of the squares in stage 1? 

TASK 1: Draw stage 2 by doing to each one of the squares in stage 1 what was done to the original square.

Q4: How many solid squares are there in stage 2? 

Q5: In comparison to the original square, what size are the squares in stage 2? 

Q6: Compared to the original, what length are the edges of the squares in stage 2?
In the diagram above your task is to understand what took place between stage 0 and stage 1 and to repeat that process on all the squares remaining in stage 1 in order to get stage 2.

Above you see stage 0 through stage 4 of the fractal known as the Sierpinski Gasket. Answer the following questions about this shape.

Q1: In comparison to the original triangle, what size are the triangles in stage 1? 

Q2: Compared to the original, what length are the edges of the triangles in stage 1? 

Q3: How many solid triangles are there in stage 3? 

Q4: In comparison to the original triangle, what size are the triangles in stage 3? 

Q5: Compared to the original, what length are the edges of the triangles in stage 3? 

Q6: How many triangles are there in stage 4? 

Q7: In comparison to the original triangle, what size the triangles in stage 4? 

Q8: Compared to the original, what length are the edges of the triangles in stage 4? 

Q9: What pattern do you see in terms of triangle size and edge size and in terms of number of triangles as you go from one stage to the next? Write out your findings, briefly, below:
Today we will be exploring how to find the perimeter and area of fractals. Images have provided for you in order to make your work easier. We will also look at ‘deconstructing’ fractals, that is we will be looking at starting with a stage 2 image and determining the initiator and generator that gave rise to it, rather than beginning with the initiator and generator and finding stage 2. This ‘deconstruction’ is actually more useful in the real world because we are typically looking at shapes and trying to find a rules to model them. Additionally we will consider non-linear fractals, what they are and how they are created. The pages included for this in-class activity are pages 35-40. With the exception of page 40 if we do not finish in class, finish these pages as homework.

CANTOR SET

LENGTH:
SIERPINSKI CARPET

stage 0
initiator

stage 1
generator

stage 2

AREA:

PERIMETER:
KOCH SNOWFLAKE CURVE

AREA:

PERIMETER:
SIERPINSKI GASKET

AREA:

PERIMETER:
DECONSTRUCTION

stage 0
initiator

stage 1
generator

stage 2

stage 0
initiator

stage 1
generator

stage 2

stage 0
initiator

stage 1
generator

41
All fractals share the same properties in terms of being the result of an infinite process of iteration, displaying detail on all levels, exhibiting self-similarity, etc. But as you can see from the images below, non-linear fractals can look quite different from the linear fractals we have been studying. These images give you an idea, but you might want to search online in order to find images like these in color. The images you see below are all taken from the fractal known as the Mandelbrot Set.

![Mandelbrot Set Images](image1.png)

Before exploring the above images further I’ll give some explanation of the process of their creation using a number-line and real numbers, so be sure to get that in your notes. For this particular image we use an iteration of an equation rather than an iteration of a shape, and we use the complex plane rather than a Cartesian coordinate system, which means we use complex numbers and not just real numbers. This may sound pretty complicated, but the process is quite simple, actually. This set, which is so complicated as to have been called more complex than the universe itself (because this mathematical shape has infinite resolution), comes from the following short equation:

\[ z = z^2 + c \]

The letters \( z \) and \( c \) stand for complex numbers. We also set \( z \) to zero to begin with, and we choose a point \( c \) in the plane to iterate. This gives us a new \( z \), which we plug in, and we continue to use our original \( c \) until we know the ‘behavior’ of the point. We’ll do some examples together to see how this works, and we’ll look at some images together to illustrate this as well.

The complex plane looks like a Cartesian coordinate system, but instead of each point representing an ordered pair, each point is a single complex number, where the horizontal axis corresponds to the real part of the number and the vertical axis corresponds to the imaginary part of the complex number.
As part of our geometry unit we are studying fractals. Our text-book presents this work differently in some ways than I would like you to approach it, so I have created the pages in this supplement to help you in your work based on my lectures. Your book remains a good support to you in this section, and you should read the related sections there. The major difference is that in the book you are asked to look at fractals at a finite stage, but I ask you to take it all the way to infinity, and, believe it or not, that is easier! Another difference is that I will be considering the first stage of a fractal to be stage zero (which is the most common approach), but your book calls it stage one. In the following three pages you will find definitions, comments, and, for reference, a re-working of what we did on the Sierpinski Gasket.

FRACTAL GEOMETRY is a dynamic branch of geometry in which shapes are created by continual change through the following of a specific rule that is applied over and over. It has been called the geometry of nature, and it has many applications in a variety of fields from science to industry to conservation to medicine to art – as we will see in class.

TERMS (for use with linear fractals):

INITIATOR: The initiator is the starting shape and is also know as ‘stage 0.’

GENERATOR: The generator is the rule to be carried out and is also known as ‘stage 1.’

The fractal shown above is known as the SIERPINSKI CARPET. Its initiator is a square, and the generator is the process of cutting each side of the square into 3 equal segments in such a way that the shape is like a tic-tac-toe board. This process results in 9 smaller squares, but then the one in the middle is removed leaving 8. This process is then done again on each of the eight remaining squares. At each stage the number of squares in the figure increases, and the size of each of these squares decreases.

At each finite stage you don’t have a true fractal, only a pseudo-fractal. A true fractal is the result of this process being carried out infinitely many times. Although we cannot physically create a true fractal, with mathematics we have the power to determine the properties of the true fractal.

Here are the things we’ll be concerned with in working with linear fractals:

- determining the self-similarity dimension of a fractal
- determining perimeters, areas, and volumes of given fractals
- constructing stage 2 of a fractal from a given initiator and generator
- finding the initiator and generator given a later stage of a fractal
PROPERTIES OF THE SIERPINSKI CARPET:

1) SELF-SIMILARITY DIMENSION: In class we discovered that the formula for finding self-similarity dimension is

\[ D = \frac{\log N}{\log P} \]

Where \( D \) is the dimension, \( N \) is the number of smaller parts similar to the original shape, and \( P \) is the number of pieces each edge is cut into. For the Sierpinski Carpet (see image on previous page), the number of pieces each edge (side) is cut into is 3 and the number of resulting smaller copies of the original is 8, so the self-similarity dimension is

\[ D = \frac{\log 8}{\log 3} = 1.8927 \]

2) PERIMETER: In order to make things as easy as possible, let’s consider the length of one side of the original square to be 1 unit. The original perimeter is then 4 units. The perimeter of a fractal is the sum of the lengths of all edges, both outside and inside, so at stage 1, we have the original 4 units still on the outside, but we have added line segments on the inside: 4 more lengths each with a measure of a third of a unit around that inner removed square, so the perimeter in units is now the original outer perimeter of four units combined with the new inner perimeter of \( 4 \times \frac{1}{3} = \frac{4}{3} \) units as shown below:

\[ 4 + 4 \times \frac{1}{3} = \frac{16}{3} \text{ or } 5 \frac{1}{3} \]

In stage 2 each of the small, newly added segments around the smallest interior squares is one-third of one-third the length of an original side – or one-ninth of a unit. We have added four small segments in each of the eight squares, so we have four times eight or 32 new segments that are each one-ninth of a unit. Therefore our perimeter in units is now

\[ 4 + 4 \times \frac{1}{3} + 32 \times \frac{1}{9} = \frac{80}{9} \text{ or } 8 \frac{8}{9} \]

The perimeter is increasing at each stage. What happens to it when we have carried out this process infinitely many times and arrived at the true fractal? In order to determine that we need to extend our series a bit further. Notice that at each stage the size of the segments is one-third their length at the previous stage, so our denominators will be 1, 3, 9, 27, 81, 243, 729, . . . Also notice that beyond the first stage we have 8 times as many squares to remove the centers from. This means that beyond stage one we will multiply by \( \frac{8}{3} \) each time to get the next number in the series.

\[ 4 + 4 \times \frac{1}{3} + 32 \times \frac{1}{9} + 256 \times \frac{1}{27} + 2048 \times \frac{1}{81} + 16384 \times \frac{1}{243} + 131072 \times \frac{1}{729} + \ldots \]

OR

\[ 4 + \left( 4 \times \frac{3}{3} + 32 \times \frac{9}{9} + 256 \times \frac{27}{27} + 2048 \times \frac{81}{81} + 16384 \times \frac{243}{243} + 131072 \times \frac{729}{729} + \ldots \right) \]

In either representation of the series we can see that the numbers we are adding each time are getting larger and larger, so the sum is infinite. Therefore this shape has an infinitely long perimeter. In the second representation, however, we can support this a little more clearly mathematically using an idea you learned in algebra. Inside of the parentheses, starting with the number \( \frac{4}{3} \), you get to the next number by multiplying by \( \frac{8}{3} \) over and over. This series inside the parentheses is a very special type of series known as a geometric series. Since the common ratio, \( \frac{8}{3} \), is greater than one, the sum is infinite. (See Appendix 2 of this text if you need a review of the algebra of this type of series.)
3) **AREA**: We work with area in a manner similar to perimeter. We start with an initial area and then add or subtract the appropriate amount as we go from stage to stage. In the case of the Sierpinski Gasket, we are removing area from one stage to the next, so we will be subtracting. Assuming again that the length of an edge at stage 0 is one unit, we begin with an area of 1 square unit. In stage one, by cutting each edge into three equal pieces, we have cut the square into 9 equal smaller squares and then removed one. Each of these small squares is therefore one-ninth the size of the original square, so at stage one we have an area in square units of

\[ 1 - \frac{1}{9} \]

At the next stage we’ve cut the eight remaining new smaller squares into ninths and removed one from each of these, so we’ve removed 8 little squares that are each a ninth of a ninth or \( \frac{1}{81} \) the size of the original square. We now have an area in square units of

\[ 1 - \frac{1}{9} - \frac{8}{81} \]

From here on out we are creating 8 new small squares each time for every little square in the previous stage, and each of these smaller squares will be one-ninth the area of each square in the previous stage, so we are multiplying the top of our fraction by 8 and the bottom by 9. The result is

\[ 1 - \frac{1}{9} - \frac{8}{81} - \frac{64}{729} - \frac{512}{6561} - \frac{4096}{59049} - \ldots \]

OR (by inserting parentheses and factoring out the negative):

\[ 1 - \left( \frac{1}{9} + \frac{8}{81} + \frac{64}{729} + \frac{512}{6561} + \frac{4096}{59049} + \ldots \right) \]

The result is a little harder to determine here than it was with perimeter. The fractions we are subtracting are getting smaller and smaller, but what is the final result? Here too geometric series comes in handy. Inside the parentheses we have an infinite geometric series. The first term is \( \frac{1}{9} \), and the common ratio is \( \frac{8}{9} \). Because the absolute value of the common ratio is less than one, we can find the sum of what is inside the parentheses. Recall from algebra that the formula in this case is

\[ S = \frac{a}{1 - r} \]

where \( S \) is the sum, \( a \) is the first term, and \( r \) is the common ratio. Ignoring the 1— that is out front for now, and only focusing on the infinite geometric series in the parentheses, we have

\[ S = \frac{\frac{1}{9}}{1 - \frac{8}{9}} = \frac{\frac{1}{9}}{\frac{1}{9}} = 1 \]

Inserting this in the place of the parentheses, we have 1 – 1, which is zero. The area is 0 square units.

**RECAP**: In the Sierpinski Carpet we have a rather unusual shape, one that has dimension 1.8927 and infinite perimeter but zero area. With most linear fractals, to find perimeter, area and/or volume, you will need to look for an infinite geometric series. Often the series will not include the first term or the first or second term. You will need to consider how much smaller the pieces are getting each time and by what ratio they are increasing or decreasing in order to find the terms in your series. It comes down to two things:

- knowing how fractions work (things like the fact that a third of a third is a ninth) and being able to use that to compare the size of the smaller pieces at each stage to the size of the original piece; this will give you the bottom of your fraction (if a geometric series is involved).
- being able count how many pieces have been added or subtracted or acted upon (depending on what type of fractal you have) from one stage to the next, and finding a pattern for this counting; this number will go on top of your fraction (if a geometric series is involved).
The image below is the Mitsubishi Gasket Fractal. Questions 14-22 refer to this image.

14) Into how many pieces was each edge of the original shape cut? 

15) How many smaller copies of the original triangle are there in stage 1 (the generator)? 

16) What size is each smaller copy in stage 1? That is, what fraction of the original triangle is each smaller triangle you see in stage 1 (the generator)? 

17) How many smaller copies of the original triangle are there in stage 2? 

18) What size is each of these smaller copies in stage 2? (That is, what fraction of the original triangle is each smaller triangle that you see in stage 2?) 

19) Based on our in-class discussion of self-similarity dimension and what it means - and the fact that the Mitsubishi Gasket begins as a 2-dimensional triangle that then has pieces removed over and over - what would you estimate the self-similarity dimension of this fractal is? 

20) Find the actual self-similarity dimension of the Mitsubishi Gasket. (HINT: Use your answers to #14 and #15 and the formula for the dimension of a fractal). How does this compare to your estimate?
21) Find the perimeter of the Mitsubishi Gasket Fractal. (HINT: Pretend each side of the original triangle has a length of one unit so that the perimeter of the original triangle is 3 units. Determine how many new segments are added to the inside at each stage and how long they are, then add the new length to the original. Do this as you go from stage to stage and then look for a geometric series to form. In looking for the series, ignore the first term of your addition. What does the ratio tell you? See Appendix 2 if you need to review the algebra of series.)

22) Find the area of the Mitsubishi Gasket Fractal. (HINT: Pretend the original triangle has an area of one square unit. Determine how much area is removed from one stage to the next by considering the size of the pieces removed and the number of the pieces removed. Look for a geometric series. In looking for the series, ignore the first term of your expression. See Appendix 2 if you need to review the algebra of series.)
23) The initiator (stage 0) and generator (stage 1) below are for the SANDERS ARROWHEAD FRACTAL. This fractal is very similar to the Sierpinski Gasket, but in this case it is the bottom left-hand square that has been removed rather than the center square. Here too there are 8 new smaller squares created as you go from initiator to generator. So, the rule is to remove the bottom left-hand one-ninth of each of the newly created smaller squares each time. Draw stage 2 of the Sanders Arrowhead Fractal. Problems 24-28 also refer to this fractal.

24) Into how many pieces was each edge of the original shape cut?

25) How many smaller copies of the original square are there in stage 1?

26) What size is each of these smaller copies in stage 1? (That is, what fraction of the original square is each smaller square that you see in stage 1?)

27) Based on our in-class discussion of self-similarity dimension and what it means - and the fact that the Sanders Arrowhead Fractal begins as a 2-dimensional square that then has pieces removed over and over - what would you estimate the self-similarity dimension of this fractal is?

28) Find the actual self-similarity dimension of the Sanders Arrowhead Fractal. How does this compare to your estimate?
Here is the MENGER SPONGE FRACTAL. It is created by beginning with a cube, dividing it up as you see below, and then removing the central cube on each face and the cube in the very center, and then, of course, repeating this process on each smaller cube at the next stage. Questions 29-36 refer to this description and to the image below.

![Menger Sponge Fractal Image]

29) Into how many pieces was each edge of the original shape cut?

30) How many smaller copies of the original cube are there in stage 1?

31) What size is each of these smaller copies in stage 1? (That is, what fraction of the original cube is each smaller cube that you see in stage 1?)

32) How many smaller copies of the original cube are there in stage 2?

33) What size is each of these smaller copies in stage 2? (That is, what fraction of the original cube is each smaller cube that you see in stage 2?)

34) Based on our in-class discussion of self-similarity dimension and what it means - and the fact that the Menger Sponge Fractal begins as a 3-dimensional square that then has pieces removed over and over - what would you estimate the self-similarity dimension of this fractal is?

35) Find the actual self-similarity dimension of the Menger Sponge Fractal. How does this compare to your estimate?
36) Find the volume of the Menger Sponge Fractal. (HINT: Use the same thought process as you did for fractals that began as 2-dimensional shapes. Use what you know about the size of the smaller copies at each stage and how many of them there are. Begin by assuming the original volume is 1 square unit, and then subtract the volume that has been removed stage-by-stage. Look for an infinite geometric series, remembering that generally the first term needs to be left in order to find the series.)

Fractals are not always created by removal (as you know by having seen the Koch Snowflake Curve in class). The fractal imaged below corresponds to questions 37-41.

37) How many smaller squares were added to the original shape?

38) What size is each of these smaller copies in stage 1? (That is, what fraction of the original square is each smaller square that you see in stage 1?)

39) Notice that a square is being added to the center of each segment each time. How many new segments are created from a previous segment at each stage?

40) How will you determine how many new squares will be added at stage 3?
41) Using a process similar to what you’ve done with previous problems of this sort find the area of the fractal pictured on the previous page.

42) Create a fractal that has a self-similarity dimension of 1.5.

**HINT:** Use our formulas AND your knowledge of properties of logarithms from algebra. Recall that

\[(\text{number of pieces a side is cut into})^\text{dimension} = \text{number of smaller copies of original}\]

That is:

\[P^D = N\]

Therefore D (dimension) is:

\[D = \frac{\log N}{\log P}\]

Find numbers that will work and then create a shape using those numbers. Use the grids below to draw stages 0-2 of your fractal. Additional hint: I recommend beginning with a shape that is a square as it is the easiest shape to work with!
43) Using the initiator and generator below, do the following four things. In a clearly organized and clearly labeled fashion, show your work in space at the bottom of this page.

(a) Draw stage 2 of this fractal on the grid provided.
(b) Find the dimension of this fractal.
(c) Find the perimeter of this fractal.
(d) Find the area of this fractal.

![stage 0](image1)
stage 0
initiator

![stage 1](image2)
stage 1
generator

![stage 2](image3)
stage 2
44) In the natural world, growth often takes place by repetition on smaller and smaller scales. Think of a tree. The trunk splits into large boughs which split into branches which split into twigs. All along the tree is growing, and at each stage the shape is similar but merely at a smaller scale. This is exactly what linear fractals do, and it is why they are so useful in modeling the natural world. In order to do find a fractal that mimics a shape in the natural world (something that is often done in movie special effects) we must work backwards from the natural fractal to an initiator and generator that will give us that result. That’s what this problem is about. Given stage 2, find the initiator and the generator.

A FEW COMMENTS ABOUT NON-LINEAR FRACTALS:

The fractals we have looked at so far fall into the category of ‘linear fractals.’ Linear fractals are those for which if you take a small piece of the final result and magnify it you will get an exact copy of the original. They display perfect self-similarity. These fractals can be created in many ways, but the method we have used of beginning with a shape and then adding or removing smaller copies of the original over and over is a very commonly used method. Non-linear fractals display self-similarity as well (all fractals do), but it is not perfect self-similarity. Magnifying a small piece will not result in an exact copy of the whole.

All fractals are the result of a process of infinite iteration. For our linear fractals we iterated a geometrical rule on a shape. Many non-linear fractals are created by iterating an equation involving complex numbers on the complex plane. The most famous of these is called the Mandelbrot Set (see image below).

There are many beautiful and interesting fractals that are created in this way, including Newton’s Method Fractal and Julia Sets. Go to youtube and search for “Mandelbrot Set Zooms.” You may also want to google these other terms to find these sorts of images online. As you watch a zoom of the Mandelbrot Set realize that the parts in black are the set itself; the points in black do not grow without bound under iteration (something I’ll explain in class), but the points that are colored get larger without bound under iteration. The point is that parts of this fractal infinitesimally close to each other behave in different ways, even though it seems like common sense to think that points close together should behave the same way. This is called sensitive dependence on initial conditions. It is something we see in chaotic processes like the
flow of turbulent fluids (water in a river) or weather. Fractals have been called pictures of chaos, and for that reason they work well for modeling and understanding many aspects of the real world that we could not model or understand with classical geometry. There are many such applications. A few of them are discussed below.

APPLICATIONS:

Fractals can be used in many fields. Fractals are frequently used in special effects in movies – the first of these being the creation of the Genesis Planet in Star Trek II: The Wrath of Khan (1982) – a more recent example being the lava scene in Star Wars III: Revenge of the Sith (2005). Fractal geometry is used in image compression; this allows you to upload images quickly on your computer or iPhone. Fractals are used in creating effective antennae for cell phones. They are used in studying the spread of disease in epidemics and in creating devices to help alleviate seizures in people with epilepsy. They are being studied for use in early prediction of cancer. The have implications for weather predictions. They’ve been used to study and decrease static in phone lines and to determine patterns in the distributions of galaxies in the universe. Your body is made up of fractals – neurons, circulatory system, pulmonary system, intestines, brain surface, etc. Much of the natural world from shapes of trees and clouds to shapes of coastlines can be modeled using fractal geometry. It has been used in conservation efforts relating to determining how much carbon dioxide a forest removes from the atmosphere. Fractal geometry has also been used in studying trends in the stock market. There are many other applications as well!

Fractal geometry is a relatively new branch of mathematics, math being a field that is always growing! Although it is rather new (1980s), it has begun, surprisingly enough for a topic in math, to show up in popular culture, especially in novels. As you would expect it has made an appearance in sci-fi/fantasy novels such as the work of Piers Anthony (Virtual Mode, Chaos Mode and Fractal Mode). It was also the impetus for the book and subsequently the movie Jurassic Park. It has even made an appearance in a recent popular Christian novel The Shack. I think as time goes on you will see it more and more in books, movies, music, etc.

By the way, from your algebra classes you have all the tools you need to understand the concepts behind non-linear fractals. If you would like to pursue this further, it is accessible by you. I am, of course, more than willing to point you in the right direction if you’d like to explore further.

45) Given the formula \( z \mapsto z^2 + c \), and based on what we did in class, **determine** whether or not the point \( c = -1 + i \) is in the set, **and** give a one sentence description explaining your answer. (**IF** - and only if - we did not get time to cover the formula for the Mandelbrot Set in class, then skip this problem.)
Read over the following questions before watching NOVA’s Fractals: Hunting the Hidden Dimension; then answer the questions as you watch or after you watch. If you need to review the video it can be found online at http://www.pbs.org/wgbh/nova/physics/hunting-hidden-dimension.html.

What are some applications of fractal geometry? (List at least 5, and be sure to use full sentences in your answer.)

What application of fractals most caught your attention, and why?

Though fractal geometry provides amazing, powerful and widely applicable results many mathematicians and scientists did not accept this new branch of mathematics at first. Why not?

Which of the 5 characteristics of all fractals that we listed in class did you see in this video? Where did you see them?
46) Below you are given the numerical values of Egyptian Numerals. Use this information and your notes on the Egyptian Multiplication Algorithm to do the multiplication problem given below the chart. Do your work using the Egyptian Method and entirely in Egyptian Numerals.

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\[ \text{𓊽} \times \text{𓊽} \]
47) Translate \(13 \times 114\) into Egyptian; work this problem using the Egyptian algorithm and give your answer in Egyptian.

48) Write out, in Mayan Numerals, the numbers 1 to 30. Use your notes, but be resourceful too and realize you can search online. Make sure, though, that you understand how to create each numeral.

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49) Write the first thirty numbers in base three.

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**Conversions of Bases**

Converting **FROM another base TO base 10** is merely a matter of using ‘expanded notation’ - of just multiplyin and adding. For instance, in the problem above asking you to write numbers in base three, you should have come up with the answer 212 as the base three representation of twenty-three. This is because the first three place values for base three are 9, 3 and 1 and because two groups of 9 plus one group of 3 plus two groups of 1 give you twenty-three (as shown below).

\[
2 \times 3^2 + 1 \times 3^1 + 2 \times 3^0 = 2 \times 9 + 1 \times 3 + 2 \times 1 = 23_{ten}
\]

Similarly, considering base five, \(1324_{five} = 214_{ten}\) because:

\[
1 \times 5^3 + 3 \times 5^2 + 2 \times 5^1 + 4 \times 5^0 = 1 \times 125 + 3 \times 25 + 2 \times 5 + 4 \times 1 = 214_{ten}
\]

This process is covered quite thoroughly in section 7.1 of your textbook. What isn’t covered there but is also part of our course is being able to convert **FROM base 10 TO another base**. As you might imagine the procedure is exactly the reverse of the procedure you see above. Instead of multiplying and adding, you divide and subtract. On the next page we consider this process using one of the examples above, and then some problems are given for you to do. (This procedure is also covered in your notes.)
We saw on the previous page that $212_{\text{three}} = 23_{\text{ten}}$ by staring with $212_{\text{three}}$ and using expanded notation. What if we had been given $23_{\text{ten}}$ and asked to convert it to base three? First we would have to consider the place values in base three: 27, 9, 3, 1. We would look for the highest value that will fit into twenty-three. In this case it is a 9. Then we see how many groups of 9 go into 23

$$23 \div 9 = 2 \text{ with a remainder of 5}$$

SO we have 2 in the 9’s place. Then we need to take care of what is left, the remainder. We move down one place to the 3’s place and ask how many 3’s there are in 5:

$$5 \div 3 = 1 \text{ with a remainder of 2}$$

SO we have 1 in the 3’s place. Then we need to take care of what is left, the remainder. We move down one place to the 1’s place and ask how many 1’s there are in 2:

$$2 \div 1 = 2 \text{ with no remainder}$$

SO we have a 2 in the 1’s place for a final result of $212_{\text{three}}$

This is the process - find the highest place value that is not larger than your number. Divide it into your number; the quotient is the number that goes in that place, and then you repeat this process with the remainder and the next place value down. Convert the following base ten numbers in to the requested places; show your work.

50) Convert forty-two into base three.

51) Convert one-hundred fifty-seven into base five.

52) Convert three-hundred ninety-eight into base sixteen.

*HINT: You can check your answers by converting back the other way through multiplication and addition! Did you get these right?!*
53) Go to the following link and try to figure out how the trick works. Explain in detail.

http://www.readthemind.com
54) Do the “Phone Number Problem” below and answer the questions that appear after the problem.

**Phone Number Problem**

1) Type into your calculator the first 3 digits of your phone number (not area code).
2) Multiply by 80 (and hit the = sign).
3) Add 1 (and hit the = sign).
4) Multiply by 250 (and hit the = sign).
5) Add the last 4 digits of your phone number (and hit the = sign).
6) Repeat step 5.
7) Subtract 250 (and hit the = sign).
8) Divide by 2 (and hit the = sign).

QUESTIONS: What is the result?
   Are you surprised?
   How does this trick work? Write out your answer in detail.
55) One million light bulbs are controlled by one million switches numbered in order from 1 to 1,000,000. All switches are in the off position to begin. Starting at 1, every switch is flipped. Next, starting at 2, every second switch is flipped. Then starting at 3, every third switch is flipped. (Of course, if a switch was off, flipping it turns it on and vice versa). This continues until the millionth switch is reached. After all this, how many light bulbs will be on?
Most or all of us probably remember some letter or number games from childhood. Perhaps you remember games from long car trips like finding all the letters of the alphabet on signs or license plates or games you might hear on the playground like “eenie-menie-minie-moe” or “I one it, I two it . . . I jumped over it and you ate it!” One such counting game has the following rules:

This is a two person game, and the winner is the person who says “21.” We start with the number 1, and each of us can count one or two or three numbers at a time.”

Just like with “eenie-menie-minie-moe” there is a way to win every time if you’re clever and set it up just right. Play this game a few times (you might want to keep a record of which numbers each player says), think about strategy as you play, and then answer the following questions:

1. What strategy will allow you to win every time?
2. How can you win this game in general even if you are counting to a number other than 21 or can count by groups of more than three numbers?
57) There are hundreds of patterns in the following triangle. Find as many patterns in it as you can (including how it is created!). Patterns may be along the rows, columns or diagonals or in different configurations. I have give you a few copies of this over the next few pages so you can try finding patterns without having to erase. You may list more than one pattern per page if you would like. At the bottom of this page I have listed a couple of the patterns and have asked some questions to get you started finding your own.

Here are a few patterns and questions to get you started:

- This triangle has horizontal symmetry.
- There are ones down the left and right sides of the triangle.
- The counting numbers can be found in the second diagonal in.
- If you choose a number in the interior, like the 4 in the fifth row down, and you look at the numbers circling it, the products of alternating numbers are equal - that is: $3 \times 10 \times 1 = 6 \times 5 \times 1$. This works for the numbers surrounding any cell in the interior.

Here are some questions that might help you in your search:

- Is there anything special about any of the other diagonals?
- What happens if you add up all the numbers in a row and look at the sums of each row?
- Can you find a pattern made up of odd numbers?
- Can you find the Fibonacci Numbers in the triangle?
- What are the first few powers of 11? Can you find them in the triangle? Do they continue all the way?

Those are just ideas to get you started. There are many, MANY patterns that can be found in this triangle! See how creative you can be in your search!
58) Twenty-four dogs are in a kennel. Twelve of the dogs are black, six of the dogs have short tails, and fifteen of the dogs have long hair. There is only one dog that is black with a short tail and long hair. Two of the dogs are black with short tails and do not have long hair. Two of the dogs have short tails and long hair but are not black. If all of the dogs in the kennel have at least one of the mentioned characteristics, how many dogs are black with long hair but do not have short tails?

59) One aspect of Set Theory is counting. The image above is of a layout of 12 of the cards out of a game called Set. Each card in the game displays 4 attributes: number, shape, shading and color. Since this workbook is not in color, you will want to go to the link below the image to get fully acquainted with the game. The object of the game is to find ‘sets.’ A set is a group of three cards for which each of the 4 characteristics is the same for all three cards or different for all three cards. For example, one set would be three cards that each have three green ovals, but each of the ovals has a different shading (solid, open and lined). Another set would be one in which all characteristics are different, for instance one card has one solid red diamond, another card has two open purple ovals, and the third card has three lined green squiggles. (Go to the link and play the game so that this will make more sense) How many cards are there in this game, and how many distinct sets of 3 cards are possible? Remember there are 3 colors, 3 shadings, 3 shapes, and 3 numbers. Please note that the number of cards in the game is not just the 12 you see above. That’s just one ‘deal.’ Also, the number of possible 3-card sets is not just the total number of cards divided by 3. Think more deeply than that. :-)

http://www.setgame.com/set/puzzle_frame.htm
60) Using your notes (the example we did together in class) as a template, prove that the set of whole numbers is infinite. For this problem use the set of natural numbers as a proper subset of the whole numbers.
61) Compare the number of points in a one-unit-long line segment (let’s say the segment on the number-line from 0 to 1) and the number of points in a cube that has length, width and height of one unit. Are there more points in one shape than the other? If so, explain using good mathematical logic why this is true. If there are the same number of points in each, explain using good mathematical logic why that is true. (Yep, this is super hard and probably really hurting your brain, as many problems in this class surely have! But remember: Nothing we study is a waste. But the precision of math helps refine how we think in a very special way. This might be a good time to go back to page 4 to remind yourself why we do this stuff! :-) 

62) In the alternate universe known as Cantor’s Paradise there are lodgings known as The Hilbert Hotel. This hotel has infinitely many rooms. If I arrive and find that all the rooms are filled, I know they can still accommodate me because they can simply have each guest move down one room, leaving room one open for me – because the person who was in room #1 is now in room #2. In mathematical terms the guest in room n moves to room n + 1. But what if the hotel is filled and a bus with infinitely many passengers pulls up? Can they all get rooms or will some of them or all of them need to be turned away? Give your answer and explain it using good mathematical logic. Notice that with both the bus and the hotel you are working with the lowest level of infinity - that of the counting numbers, not the reals. 

(Please note the answer of moving everyone 1 room over with the n + 1 rule given above is for ONE PERSON! The answer to moving infinitely more people in is not to move one by one by one like that. How can you move infinitely people in with one application of a single different rule applied once?)
63) Problem solving is an activity that requires logic. You’ve done lots of that in this class. For this problem, now that we have had a lesson on induction and deduction, revisit problems 53-57 and determine if your work on them involved inductive or deductive reasoning.

53 was ____________________________

54 was ____________________________

55 was ____________________________

56 was ____________________________

57 was ____________________________

64) How many people are in class today? Write your answer in this blank, so you don’t forget later ______. If everyone in class today were to shake hands with everyone else in class (one-handshake per pair of people), how many handshakes would there be total?
In the lesson on logical statements we saw that self-reference, such as “This sentence is false” can be a paradoxical problem. However, as we shall see here, it can also be helpful in creating and solving logic puzzles. The following puzzles have been taken from *The Lady or the Tiger? And Other Logic Puzzles: Including a Mathematical Novel and Gödel’s Great Discovery* by Raymond Smullyan. How’s that for a book title? Smullyan got his inspiration for these particular puzzles from the short story *The Lady or the Tiger* by Frank Stockton - a good read, which you might want to check out if you haven’t already.

According to Smullyan, the king of a certain land had also read Stockton’s story and decided it was a perfect way to try his prisoners. He would give each prisoner a choice of two doors behind which could be a beautiful young lady (whom he would then marry if he chose that door), or a hungry tiger (by whom he would be eaten if he chose that door). Rather than leaving it to random chance, the king decided to post signs on the doors and to give the prisoner certain facts about those signs.

**TRIALS DAY ONE:**

As each prisoner was brought out the king explained to him that each of the two rooms he was facing contained either a lady or a tiger, but that it could be that there were tigers in both rooms or ladies in both rooms.

A) As the FIRST PRISONER faced the two doors he was told of the signs below, “One of them is true, but the other one is false.” If you were the prisoner, which door would you open to find the lady?

I

In this room there is a lady, and in the other room there is a tiger.

II

In one of these rooms there is a lady, and in one of these rooms there is a tiger.
B) The first prisoner solved the puzzle, thus saving his own life. The signs on the doors were then changed, and new occupants for each room were chosen. This time the signs read as follows. The kind told the SECOND PRISONER that the signs were either both true or both false. What should the he do if he wishes to remain alive?

\begin{figure}
\centering
\begin{tabular}{|c|}
\hline
\textbf{I} & \textbf{II} \\
\hline
At least one of these rooms contains a lady. & A tiger is in the other room. \\
\hline
\end{tabular}
\end{figure}

TRIALS DAY TWO:

Both prisoners on the first day were able to solve the puzzles. The king considered this a fiasco and decided to make the puzzles harder for the next day. The king explained to each prisoner that in the left-hand room (Room I), if a lady is in it, then the sign on the door is true, but if a tiger is in it, then the sign is false. In the right-hand room (Room II), the situation is the opposite: a lady in the room means the sign on the door is false, and a tiger in the room means the sign is true. Again, it is possible that both rooms contain ladies or both rooms contain tigers, or that one room contains a lady and the other a tiger.

C) The THIRD PRISONER faced the two doors with signs shown below, and knowing the information above, which choice should he make to save his life?

\begin{figure}
\centering
\begin{tabular}{|c|}
\hline
\textbf{I} & \textbf{II} \\
\hline
Both rooms contain ladies. & Both rooms contain ladies. \\
\hline
\end{tabular}
\end{figure}
D) Same rules apply for the FOURTH PRISONER. Below is what he sees. What should he do?

I
At least one room contains a lady.

II
The other room contains a lady.

E) The king was particularly fond of the previous puzzle and this one too! What should PRISONER FIVE do?

I
It makes no difference which room you pick

II
There is a lady in the other room.

F) Here are the signs for the SIXTH PRISONER. What should he do?

I
It does make a difference which room you pick

II
You are better off choosing the other room.
Appendix 1: 30-60-90 Triangle

When working with networks we need to be able to find lengths of sides of “30-60-90 triangles,” that is triangles whose angle measures are 30 degrees, 60 degrees and 90 degrees. We can find the necessary formulas by doing two things:

1. considering a regular (equilateral) triangle and cutting it in half
2. using the Pythagorean Theorem, $a^2 + b^2 = c^2$, to find side-lengths

Recall that the total number of degrees in a triangle is 180°. Therefore, an equilateral triangle (all sides equal, all angles equal) has angles that are all 60°. If we cut an equilateral triangle in half by drawing an altitude (a line that meets the base at a 90° angle), the result will be a 30-60-90 triangle, because on of the 60° angles will remain untouched, the other will be cut in half, forming a 30° angle, and at the base we will have a 90° angle because of the perpendicular altitude. See diagram below:

Two of the sides are easy to find the measure of. The hypotenuse (longest side) of the 30-60-90 triangle is length $s$ (the length of the side of the original equilateral triangle). The length of the shortest side is half of $s$ (because we cut it in half!). We have to use the Pythagorean Theorem to find the length of the other side. When we do that by letting $c$ be $s$ and letting $a$ be $\frac{1}{2}s$, we get that side $b$ is $\frac{\sqrt{3}}{2}s$.

It’s easier to find values of certain sides. For instance, if you know the hypotenuse (longest side) is 6 inches, then you know the shortest side is 3 inches, because it’s half. The “medium-length” side is 6 times $\frac{\sqrt{3}}{2}s$, which is $3\sqrt{3}$, which is just a bit trickier to find. The hardest situation is when you know the measure of the “medium-length” side and have to find the others. Realize, though, that whatever the length of that side is, you just set it equal to it’s value, which is $\frac{\sqrt{3}}{2}s$, and solve for $s$ by getting the variable by itself. For instance, if the “medium-length” side is 10 inches, then:

$$10 = \frac{\sqrt{3}}{2}s \quad \text{SO (by multiplying by reciprocal on both sides)} \quad \frac{2}{\sqrt{3}} \times 10 = s \quad \text{AND} \quad \frac{20}{\sqrt{3}} = s$$
Appendix 2: Infinite Geometric Series

In algebra we look at sequences and series. The two we focus on most are algebraic and geometric. In this class the most important of these to us will be the infinite geometric series. An infinite geometric series is a sum of terms that have a common ratio. For example:

An infinite geometric series with first term 3 and common ratio (multiplier) 2 would be:

\[ 3 + 6 + 12 + 24 + 48 + 96 + 192 + 384 + \cdots \]

An infinite geometric series with first term 2 and common ratio (multiplier) \( \frac{4}{3} \) would be:

\[ 2 + \frac{8}{3} + \frac{32}{9} + \frac{128}{27} + \frac{512}{81} + \frac{1048}{243} + \cdots \]

An infinite geometric series with first term 1 and common ratio (multiplier) \( \frac{1}{2} \) would be:

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots \]

It’s important to know the ratio. If it’s not immediately obvious, divide a term by the one that came just before it (or think, “What do I have to multiply this term by to get that next one?”).

What we’ll be especially concerned with in class is the actual sum of the series. If you think about it a minute, you’ll realize that something like the first series up there gets bigger and bigger and bigger, and if we added all the terms (infinitely many bigger and bigger terms), we’d get an infinite result. This is also the case with the second example above, because the ratio is greater than one. However, with the third example we get a finite sum, and that sum is 2

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots = 2 \]

The formula for finding this sum in general is:

\[ S = \frac{a}{1 - r} \]

where \( S \) stands for the sum, \( a \) stands for the first term, and \( r \) stands for the common ratio. This formula can only be used if \(|r| < 1\) (remember if the ratio is one or greater the sum is infinite). Below this formula is used to show that the sum of the previous series is 2

\[ a = 1, \ r = \frac{1}{2} \] so \( S = \frac{a}{1 - r} \) becomes \( S = \frac{1}{1 - \frac{1}{2}} \) which equals \( \frac{1}{\frac{1}{2}} \) which is \( 1 \cdot \frac{1}{2} \) which is \( 1 \times 2 \times 1 = 2 \)

Another example:

\[ 5 + \frac{15}{4} + \frac{45}{16} + \frac{135}{64} + \frac{405}{256} + \cdots \]

Here we have:

\[ a = 5, \ r = \frac{3}{4} \] so \( S = \frac{a}{1 - r} \) becomes \( S = \frac{5}{1 - \frac{3}{4}} \) which equals \( \frac{5}{\frac{1}{4}} \) which is \( 5 \cdot \frac{1}{4} \) which is \( 5 \times 4 \times \frac{1}{1} = 20 \]
In some of the problems we will work with in this class you will have an expression that has an infinite geometric series in it but for which some of the terms are not part of the infinite geometric series. For example:

\[
\frac{1}{3} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots
\]

in the series above we have a common ratio of $\frac{1}{2}$ from the second term on, but you can’t multiply the first term, $\frac{1}{3}$, by that ratio to get the second term, $\frac{1}{4}$, so the $\frac{1}{3}$ is not part of the geometric series. We can use parentheses to split this up, however, so that we separate out the term or terms that are not part of the geometric series:

\[
\frac{1}{3} + \left( \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots \right)
\]

To find the sum here we then use the formula on the terms in the parentheses and add the extra $\frac{1}{3}$ to that when we are done. In this case $a = \frac{1}{4}$ and $r = \frac{1}{2}$, so we have

\[
\frac{1}{3} + \left( \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots \right) = \frac{1}{3} + \frac{\frac{1}{4}}{1 - \frac{1}{2}} = \frac{1}{3} + \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{3} + \frac{1}{4} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}
\]

In many examples you will work with subtraction will be involved. Don’t forget that when you factor a negative out of parentheses it changes the sign of all the terms in side, so for the series below we see that the common ratio is $\frac{1}{5}$ and that we get from one term to the next by multiplying by this common ratio with the exception of the term 1 out front.

\[
1 - \frac{2}{3} - \frac{2}{15} - \frac{2}{75} - \frac{2}{375} - \frac{2}{1875} - \cdots
\]

We need to rewrite this as:

\[
1 - \left( \frac{2}{3} + \frac{2}{15} + \frac{2}{75} + \frac{2}{375} + \frac{2}{1875} + \cdots \right)
\]

Notice how all the signs inside the parentheses changed from minus to plus. Notice also that in parentheses we have an infinite geometric series with $a = \frac{2}{3}$ and $r = \frac{1}{5}$. Here is how we find the sum:

\[
1 - \left( \frac{2}{3} + \frac{2}{15} + \frac{2}{75} + \frac{2}{375} + \frac{2}{1875} + \cdots \right) = 1 - \frac{\frac{2}{3}}{1 - \frac{1}{5}} = 1 - \frac{\frac{2}{3}}{\frac{4}{5}} = 1 - \frac{8}{15} = \frac{7}{15}
\]

REMINDERS: Sometimes you need to leave more than one term out, so be sure you can recognize a geometric series (you need to be able to consistently multiply by the same thing over and over from one term to the next). Remember also that you cannot use the formula unless $|r| < 1$. Sometimes this actually makes things easier. If you are multiplying by a ratio that is greater than 1, then you immediately know the sum is infinite, and you can just state that without doing any more math on that problem!!

Examples of this are worked out in the context of fractals in this text on pages 30 and 31 where we are looking at the perimeter and area of the Sierpinski Carpet.
Appendix 3: Complex Numbers

For most of your work in algebra, if you had an equation like $x^2 = -1$ you would write that there is no solution because you cannot take the square root of a negative number. At some point, however, you were told this isn’t entirely true. It isn’t that there’s no solution, it’s that there’s no real solution - and there is a difference! Real numbers are the numbers you find on the number-line, but they aren’t the only numbers there are.

One solution to $x^2 = -1$ is that $x = i$ where $i$ is called the “imaginary unit” and is equal to $\sqrt{-1}$.

SO:

\[ i = \sqrt{-1} \]

AND THEREFORE:

\[ i^2 = -1 \]

There is nothing less “real” or more “imaginary” about this number than there is about numbers such as 5 and $\frac{1}{2}$. The labels are rather unfortunate, but . . . ah well . . . The “real” numbers you’ve worked with most of your life are really only part of the story. All of them are actually complex numbers, but you just usually leave off the imaginary part. That is:

- 5 is the same thing as $5 + 0i$

and

- $\frac{-3}{4}$ is the same thing as $\frac{-3}{4} + 0i$

So the imaginary part has always been there; it’s just usually invisible. As stated above, there is nothing less “real” or important about complex or imaginary numbers. They are used in many applications, including electrical engineering and “rocket science” and movie animation and the study of fluid flow and many other things. You’ve come across them before when you’ve used the quadratic formula to solve quadratics such as $x^2 + 4x + 13$.

Arithmetic with complex numbers works very much like operations on binomials, except that you have to remember that $i^2 = -1$. For instance:

Just as $(x + 2) + (3x - 5) = 4x - 3$ so also $(i + 2) + (3i - 5) = 4i - 3$

You have to be a little more careful with multiplication because of the whole $i^2 = -1$ thing:

Just as $(x+2)(x+3) = x^2+5x+6$ so also $(i+2)(i+3) = i^2+5i+6$ but this is $-1+5i+6$ which is $5i+5$

Technically there are some bigger issues to worry about when dividing, but we’re not going to divide complex numbers in this class, so don’t worry about it. (If you’re curious it has to do with rationalizing the denominator, i.e. not letting there be an $i$ in the bottom in your answer.)

In your algebra work you’ve done a lot of graphing - putting points on number-lines and putting points on the x-y plane. You’ve never come across an imaginary or complex number on either of those items. Number-lines and the x-y plane consist only of real-number coordinates. But that doesn’t mean you can’t graph complex numbers. They are graphed on a plane, but a complex plane rather than the standard x-y plane. The vertical axis is the imaginary axis, and the horizontal axis is the real axis. So to graph $3 + 2i$, you go to the right 3 and up 2 and plot the point. Realize that unlike with the Cartesian coordinate system (x-y plane) that you are used to, the point on the complex plane represents one number instead of two. This is shown on the next page.
Below is a complex plane (also known as a Gaussian Plane or Argand Diagram). The vertical axis is the imaginary axis, and the horizontal axis is the real axis. The point plotted here is the one we were discussing on the previous page.

\[ z = z^2 + c \]

The value of \( z \) always starts out at zero, and \( c \) is a point you have chosen on the plane. Square \( z \), add \( c \), and you’re result is the new \( z \). Plug it back in square \( z \), add \( c \), and you’re result is the new \( z \). Plug it back in square \( z \), add \( c \), and you’re result is the new \( z \). ETC! If the process stays bounded, color the \( c \) you used black, if not color it something other than black. Again, that’s just a start, but there’s lots of info on the internet, and I’d be happy to work with you too!