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3.2 Nondecimal Positional Systems

In the preceding section, we discussed several numeration systems, including the decimal system in common use today. As already mentioned, the decimal system is a positional system based on 10. This is probably so because we have ten fingers and historically, as now, people often counted on their fingers. In the Did You Know box at the end of this section, we will give applications of positional systems that are based on whole numbers other than 10. One very useful application of bases other than ten is as a way to deepen our understanding of number systems, arithmetic in general, and our own decimal system. Although bases other than ten were in the K–5 curriculum in the past, they are not present currently. However, nondecimal positional systems are studied in the "math methods" course that is a part of an elementary education degree and may return to elementary school in the future. We will discuss addition and subtraction in base five in the next section and multiplication in base six in the following one.

Base Five Notation

For our purposes, perhaps the quickest route to understanding **base five notation** is to reconsider the abacus of Figure 3.1. This time, however, we allow only 5 beads to be moved forward on each wire. As before, we start with the wire to the students' right (our left) and move 1 bead forward each time as we count 1, 2, 3, and so on. When we reach 5, we have counted the first wire once, and we record this on the abacus by moving 1 bead forward on the second wire while moving all 5 beads on the first wire to the back. We continue to count, and when the count reaches 10, we will have counted all the beads on the first wire a second time. We move a second bead forward on the second wire and again move all the beads on the first wire to the back. Thus, each bead on the second wire counts for 5; that is, the second wire is the "fives" wire. When the count reaches 19, the abacus will appear as in Figure 3.8, and

$$19 = 3 \cdot 5 + 4.$$

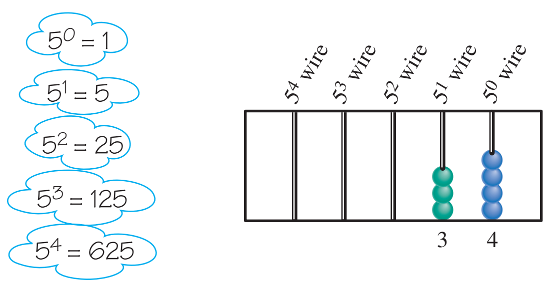

FIGURE 3.8
A count of 19 on a five-bead abacus

If we continue the count to 25, all the beads on the first wire will have been counted five times. This means that we have moved all 5 beads to the front on the *second* wire. This is recorded on the abacus by returning these beads to the back of the abacus and moving 1 bead forward on the *third* wire. Thus, the third wire becomes the $25 = 5^2$ wire. The count can be continued in this way, and it is apparent that any whole number will eventually be counted and can be recorded on the abacus with only 0, 1, 2, 3, or 4 beads per wire

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showing at the front; that is, we need only the digits 0, 1, 2, 3, and 4 in base five. For example, if we continue the counting process up to 113, the abacus will have 3 beads on the units wire, 2 on the fives wire, and 4 on the twenty-fives wire—that is,

$$113 = 4 \cdot 5^2 + 2 \cdot 5 + 3.$$

Just as we shorten $4 \cdot 10^2 + 2 \cdot 10 + 3$ to 423, someone working in base five also shortens $4 \cdot 5^2 + 2 \cdot 5 + 3$ to 423. Thus, the three-digit sequence 423 can represent many different numbers depending on which base is chosen. To the average person, who doesn't even think about it, it means "four hundred twenty-three." To our base five friend it could mean *flug globs zeit-tab*—that is, $4 \cdot 5^2 + 2 \cdot 5 + 3$. Note that our base five friend certainly would *not* say "one hundred thirteen" since that is decimal, or base ten, language. To avoid confusion in talking about base five numeration, we agree that we will *not* say "four hundred twenty-three" when we read 423 as a base five numeral. Instead, we will say "four two three base five" and will write 423_{five}, where the subscript "five" indicates the base.

In base six, we understand that

$$423_{\text{six}} = 4 \cdot 6^2 + 2 \cdot 6 + 3,$$

and we read the numeral as "four two three base six." Doing that arithmetic, it becomes clear that $423_{\text{six}} = 159_{\text{ten}}$. As an additional example, 423_{twelve} should be read "four two three base twelve," and in expanded form, we have

$$423_{\text{twelve}} = 4 \cdot 12^2 + 2 \cdot 12 + 3 = 603_{\text{ten}}.$$

Unless expressly stated to the contrary, a numeral written *without* a subscript should be read as a base ten numeral.

Observe that, in base five, we need the digits 0, 1, 2, 3, and 4, and we need to know the values of the positions in base five. In base six, the digits are 0, 1, 2, 3, 4, and 5, and we need to know the positional values in base six. In base twelve, we use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, and E, where T and E are, respectively, the digits for 10 and 11. For these bases, the positional values are given in Table 3.7.

TABLE 3.7 **Digits and Positional Values in Bases Five, Six, and Twelve**

Base b	Digits used		DECIMAL VALUES OF POSITIONS				
		. . .	b^4	b^3	b^2	b^1	units ($b^0 = 1$)
Five	0, 1, 2, 3, 4	. . .	$5^4 = 625$	$5^3 = 125$	$5^2 = 25$	$5^1 = 5$	$5^0 = 1$
Six	0, 1, 2, 3, 4, 5	. . .	$6^4 = 1296$	$6^3 = 216$	$6^2 = 36$	$6^1 = 6$	$6^0 = 1$
Twelve	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E	. . .	$12^4 = 20,736$	$12^3 = 1728$	$12^2 = 144$	$12^1 = 12$	$12^0 = 1$

EXAMPLE 3.2 Converting from Base Five to Base Ten Notation

Write the base ten representation of 3214_{five}.

Solution We use the place values of Table 3.7 along with the digit values. Thus,

$$\begin{aligned} 3214_{\text{five}} &= 3 \cdot 5^3 + 2 \cdot 5^2 + 1 \cdot 5^1 + 4 \cdot 5^0 \\ &= 3 \cdot 125 + 2 \cdot 25 + 1 \cdot 5 + 4 \cdot 1 \end{aligned}$$

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$$= 375 + 50 + 5 + 4$$
$$= 434.$$

Therefore, "three two one four base five" is four hundred thirty-four.

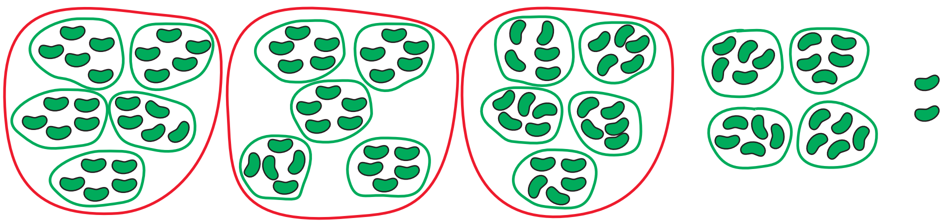
EXAMPLE 3.3 Converting from Base Ten to Base Five Notation

Write the base five representation of 97.

Solution

Recall that 97 without a subscript has its usual meaning as a base ten numeral.

We begin with the notion of grouping. Suppose that we have 97 beans and want to put them into groups of single beans (units), groups of 5 beans (fives), groups of five groups of 5 beans each (twenty-fives), and so on. The problem is to complete the grouping using the least possible number of groups. This means that we must use as many of the larger groups as possible. Diagrammatically, we have



so $97_{\text{ten}} = 342_{\text{five}}$. Arithmetically, this corresponds to determining how many of each position value in base five are required to represent 97. This can be determined by successive divisions. Referring to Table 3.7 for position values, we see that 125 is too big. Thus, we begin with 25 and divide successive remainders by successively lower position values:

$$\begin{array}{r} 3 \\ 25\overline{)97} \\ \underline{75} \\ 22 \end{array} \qquad \begin{array}{r} 4 \\ 5\overline{)22} \\ \underline{20} \\ 2 \end{array} \qquad \begin{array}{r} 2 \\ 1\overline{)2} \\ \underline{2} \\ 0 \end{array}$$

These divisions reveal that we need three 25s, four 5s, and two 1s, or, in tabular form,

25s	5s	1s
3	4	2

Thus,

$$97_{\text{ten}} = 342_{\text{five}}.$$

As a check, we note that

$$342_{\text{five}} = 3 \cdot 25 + 4 \cdot 5 + 2$$
$$= 75 + 20 + 2$$
$$= 97_{\text{ten}}.$$

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Did You Know?

The Utility of Other Bases

At first thought, it might appear that positional numeration systems in bases other than base ten are merely interesting diversions. Quite to the contrary, they are extremely important in today's society. This is particularly true of the base two, or binary, system; the base eight, or octal, system; and the base sixteen, or hexadecimal, system. The degree to which calculators and computers have affected modern life is simply enormous, and the basic notion that allows these devices to operate is base two arithmetic. A switch is either on or off; a spot on a magnetic grid is either positively or negatively magnetized; either a spot on a compact disc is activated or it is not. All of these devices are capable of recording two states—either 0 or 1—and so can be programmed to do base two arithmetic and to record other data in code "words" consisting of strings of 0s and 1s. A drawback of base two notation is that numerals for relatively small numbers become quite long. Thus,

$$60_{\text{ten}} = 111100_{\text{two}},$$

and this makes programming a computer somewhat cumbersome. The notation is greatly simplified by using octal or hexadecimal notation, since these are intimately related to binary notation. Thus, triples of binary digits become single octal digits and vice versa, and quadruples of digits in binary notation correspond to single digits in hexadecimal. For example, since

$$111_{\text{two}} = 7 \text{ and } 100_{\text{two}} = 4,$$

it follows that

$$60_{\text{ten}} = 111100_{\text{two}} = 74_{\text{eight}}.$$

For this reason, computer programmers often work in octal or hexadecimal notation.

Problem Set 3.2

Understanding Concepts

1. In a long column, write the base five numerals for the numbers from 0 through 25.

2. Briefly describe the pattern or patterns you observe in the list of numerals in problem 1.

3. Here are the base six representations of the numbers from 0 through 35 arranged in a rectangular array. Briefly describe any patterns you observe in this array.

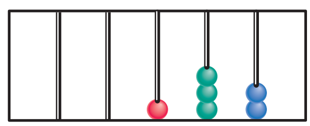
0	1	2	3	4	5
10	11	12	13	14	15
20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

4. What would be the entries in the next two rows of the table in problem 3?

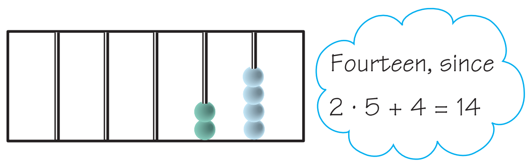
5. Write the base ten representations of each of the following:

(a) 413_{five} **(b)** 2004_{five} **(c)** 10_{five}
(d) 100_{five} **(e)** 1000_{five} **(f)** 2134_{five}

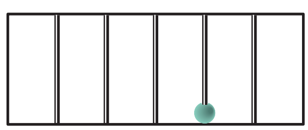
6. The abacus of Figure 3.8 has "five" playing the role of "ten." Thus 42 would be represented on the abacus as shown here. Moreover, it would be recorded as 132 and read as "one three two" and *not* as "one hundred thirty-two." What numbers would be represented if the beads on the abacus described are as shown in these diagrams?



(a)



(b)



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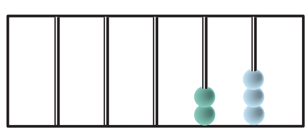

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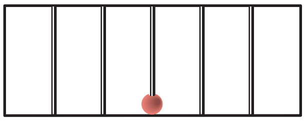
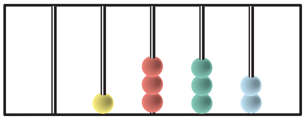
(d)

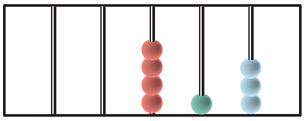


(e)



7. The number represented in the diagram of problem 6b would be recorded as 10 ("one zero"). How would the numbers for parts (a), (c), (d), and (e) naturally be recorded?

8. Suppose the abacus of problem 6 is configured as shown here.



(a) What number does this arrangement represent?

(b) How would this number naturally be recorded?

(c) Suppose the number shown in part (a) is increased by 7. Draw a diagram to show how the abacus will then be configured.

9. Write the base ten representations of each of the following:

(a) 413_{six} **(b)** 2004_{six} **(c)** 10_{six}
(d) 100_{six} **(e)** 1000_{six} **(f)** 2134_{six}

10. Write the base ten representations of each of these numbers. Remember that in base twelve, the symbols for the digits 10 and 11 are T and E, respectively.

(a) 413_{twelve} **(b)** 2004_{twelve} **(c)** 10_{twelve}
(d) 100_{twelve} **(e)** 1000_{twelve} **(f)** $2TE4_{twelve}$

11. Determine the base five representation for each of the following. Remember that a numeral with no subscript is understood to be in base ten.

(a) 362 **(b)** 27 **(c)** 5 **(d)** 25

12. Determine the base six representation for each of the following:

(a) 342 **(b)** 21 **(c)** 6 **(d)** 216

13. Determine the base twelve representation for each of the following:

(a) 2743 **(b)** 563 **(c)** 144 **(d)** 1584

14. Base two is a very useful base. Since it requires only two digits, 0 and 1, it is the system on which all calculators and computers are based.

(a) Make a table of position values for base two up as far as $2^{10} = 1024$.

(b) Write each of these in base ten notation.
(i) 1101_{two} **(ii)** 111_{two} **(iii)** 1000_{two}
(iv) 10101_{two}

(c) Write each of these in base two notation.
(i) 24 **(ii)** 18 **(iii)** 2 **(iv)** 8

(d) Write the numbers from 0 to 31 in base two notation in a vertical column, and discuss any pattern you observe in a short paragraph.

(e) In three or four sentences, compare part (d) of this problem with the Hands On activity at the beginning of the chapter.

15. **(a)** How can you use the five-bead classroom abacus to convince your students that every whole number can be represented *uniquely* (that is, in one and only one way) in base five notation?

(b) How would you modify the drawing in Figure 3.8 to illustrate the base five representation of 3241_{five}?

Teaching Concepts

16. Explain how you would respond to one of your students who claims that the base five representation of 188_{ten} is 723_{five}. Note that $7 \cdot 5^2 + 2 \cdot 5 + 3 = 188_{ten}$.

17. **(a)** In trying to help your students to better understand positional notation, you might ask them what sort of number system a tribe of natives who counted on both their fingers and toes might have invented.

(b) If your students had trouble answering the question in part (a), what might you do to help?

Thinking Critically

18. Here's an interesting trick. Consider these pictures of cards you might make for use in your class.

1	3	5	7	9	11
13	15	17	19	21	
23	25	27	29	31	

2	3	6	7	10	11
14	15	18	19	22	
23	26	27	30	31	

4	5	6	7	12	13
14	15	20	21	22	
23	28	29	30	31	

8	9	10	11	12	13
14	15	24	25	26	
27	28	29	30	31	

16	17	18	19	20	21
22	23	24	25	26	
27	28	29	30	31	

(a) Record the first number of each card that has the day of the month on which you were born.

(b) Add the numbers in part (a).

(c) Surprised? Our experience is that elementary school students are, too, and that they immediately want to know how the trick works. See if you can discover the secret by carefully comparing the cards with your answer to problem 14, part (d).

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19. **(a)** Add $11{,}111{,}111_{\text{two}}$, and 1_{two} in base two.

(b) What are the base two numerals for 2^n and $2^n - 1$? Explain briefly, describing a pattern.

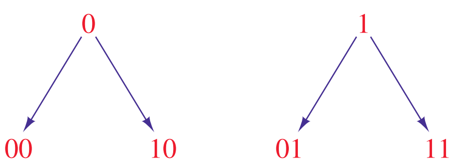
20. Recall that the rows of Pascal's triangle (see Section 1.4) are numbered 0, 1, 2, 3, ... Thus, row five is 1, 5, 10, 10, 5, 1, which has four odd entries. Also, the base two representation of 5 is 101_{two}, with two 1s and $2^2 = 4$. Remarkably, if f is the number of 1s in the base two representation of n, then 2^f gives the number of odd entries in the nth row of Pascal's triangle. Verify that this is true for the following rows of Pascal's triangle.

(a) row $n = 7$, whose entries are 1, 7, 21, 35, 35, 21, 7, 1

(b) row $n = 8$, whose entries are 1, 8, 28, 56, 70, 56, 28, 8, 1

(c) row $n = 11$, whose entries are 1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1

21. The two one-digit sequences of 0s and 1s are 0 and 1. The four two-digit sequences of 0s and 1s are 00, 10, 01, and 11.



(a) Write down all of the three-digit sequences of 0s and 1s. Can you think of an easy way to do this? Explain.

(b) Write down all of the four-digit sequences of 0s and 1s. Can you think of an easy way to do this? Explain.

(c) How many five-digit sequences of 0s and 1s are there?

HM **(d)** How many n-digit sequences of 0s and 1s are there? Explain why this is so.

22. **(a)** Think of each three-digit sequence of 0s and 1s in problem 21a as a base two numeral. What are the base ten equivalents of these numerals?

$$011_{\text{two}} = 0 \cdot 2^2 + 1 \cdot 2 + 1$$
$$= 3_{\text{ten}}$$

(b) Think of each four-digit sequence of 0s and 1s in problem 21b as a base two numeral. What are the base ten equivalents of these numerals?

(c) Think of each n-digit sequence of 0s and 1s as a base two numeral. What do you think are the base ten equivalents of these numerals? Explain.

Using a Calculator

23. **(a)** Lyudmila discovered a method to convert a numeral in any base to its equivalent in base ten using her calculator. For example, to convert 423_{five}, she used the entry string

4 [×] 5 [+] 2 [=] [×] 5 [+] 3 [=]

to obtain the correct result of 113. Similarly, to convert 4075_{eight}, she used the string

4 [×] 8 [+] 0 [=] [×] 8 [+] 7 [=]
[×] 8 [+] 5 [=]

to obtain the correct result of 2109. Lyudmila isn't quite sure why her procedure works. Write an imaginary dialogue with her to help her understand why the method is valid.

(b) Would this method succeed on an ordinary four-function calculator?

For Review

24. Write two division equations corresponding to each of these multiplication equations.

(a) $3 \cdot 17 = 51$ **(b)** $11 \cdot 91 = 1001$

(c) $9 \cdot 121 = 1089$

25. Write a multiplication equation and a second division equation corresponding to each of these division equations.

(a) $341 \div 11 = 31$ **(b)** $455 \div 65 = 7$

(c) $124{,}857 \div 13{,}873 = 9$

26. Lida Lee has three sweaters, four blouses, and two pairs of slacks that mix and match beautifully. How many different outfits can she wear using these nine items of clothing?

27. **(a)** If n represents Shane Stagg's age six years ago, how old is Shane now?

(b) If m represents Shane's age now, how old was Shane six years ago?

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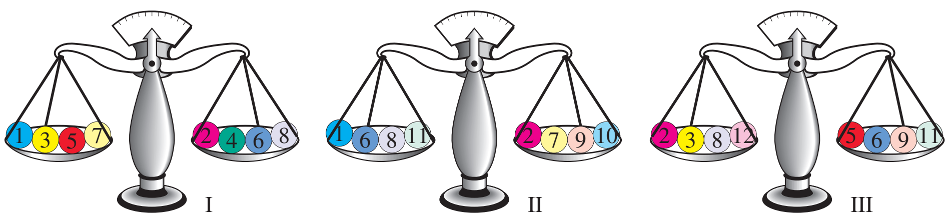
Cooperative Investigation

A Remarkable Base Three Trick

Practice performing this trick with a partner until you both can do it with ease.

Mathematical magic certainly has a place in the classroom. Students find it interesting, fun, and highly motivational. Consider the following problem: You are given 12 coins that appear identical, but one of the coins is false and is either too heavy or too light—you don't know which. You have a balance scale and are to find the false coin and to determine whether it is heavy or light in just three weighings. This is a difficult problem that is usually solved by considering a whole series of cases. However, if you know base three arithmetic, you can determine the false coin so quickly right in front of your class that you appear to be a real wizard! Here's how it can be done.

1. Number the coins from 1 through 12.
2. With your back turned, ask your class (partner) to agree on the number of the coin it (he or she) wants to be false and to decide whether it is to be heavy or light.
3. Indicate that you are going to make three weighings, and ask the class (your partner) what the movement of the *left-hand* pan on each weighing will be. If the left-hand pan goes up, record a 2. If it balances, record a 1. If it goes down, record a 0. Now use the results of these weighings to form a three-digit base three numeral. The first weighing determines the *nines* digit, the second weighing determines the *threes* digit, and the third weighing determines the *units* digit. If the base ten number determined in this way is less than 13, it names the false coin. If the number is more than 13, then 26 minus the number generated names the false coin. You can tell, by knowing which is the false coin, whether it is heavy or light by noting how the left-hand pan moved on a weighing involving the false coin.



Note the addition using base three notation,

$$\begin{array}{r} 201 \\ +021 \\ \hline 222 \end{array}$$

and $222_{\text{three}} = 26_{\text{ten}}$

For example, suppose the class (your partner) chooses coin 7 as the false coin and decides that it should be heavy. Then on the three weighings, the left-hand pan goes down, goes up, and balances, and we generate $021_{\text{three}} = 7$, the number of the false coin. Moreover, since coin 7 was in the left-hand pan, which went down on the first weighing, coin 7 must be heavy. On the other hand, suppose that coin 7 is light. Then on the three weighings, the left-hand pan goes up, goes down, and balances, and we generate $201_{\text{three}} = 19$. But then $26 - 19 = 7$, so coin 7 is the false coin and must be light since the left-hand pan went up on the first weighing.

Our experience is that children like the trick very much and are strongly motivated to learn base three notation in order to pull the trick on their friends and parents. Note also that, to do the trick well, students must be able to perform mental calculations quickly—itself a worthy goal.

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